Integration by u-Substitution

aka Reversing Chain Rules

u-Substitution: Reversing the Power Chain Rule

$$D[f(x)^n] = n\, f(x)^{n-1} D[f(x)]$$

$$D\left[\frac{f(x)^n}{n}\right] = f(x)^{n-1} D[f(x)]$$

$$D\left[\frac{f(x)^{n+1}}{n+1}\right] = f(x)^n\, D[f(x)]$$

$$\frac{f(x)^{n+1}}{n+1} + C = \int f(x)^n\, D[f(x)]\, dx$$

Let $u = f(x)$, $du = D[f(x)]\, dx$, and then substitute:

$$\frac{u^{n+1}}{n+1} + C = \int u^n\, du$$

Example 1. u-Substitution: Reversing the Power Chain Rule

$$D[(x^2 + 1)^3] = 3(x^2 + 1)^2 D[(x^2 + 1)]$$

$$D\left[\frac{(x^2 + 1)^3}{3}\right] = (x^2 + 1)^2 \, 2x$$

$$\frac{(x^2 + 1)^3}{3} + C = \int (x^2 + 1)^2 \, 2x \, dx$$

Let $u = (x^2 + 1)$, $du = 2x\,dx$, and then the pattern above is:

$$\frac{u^{n+1}}{n+1} + C = \int u^n \, du$$

Example 2. u-Substitution: Reversing the Power Chain Rule

Suppose we want to find the following antiderivative:

$$\int (x^2 + 1)^2 \, x \, dx$$

Then we can use the following pattern:

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

Let $u = (x^2 + 1)$, $du = 2x\,dx$, and then try to match the pattern above:

$$\frac{1}{2}\int (x^2 + 1)^2 \, 2\,x\,dx = \frac{1}{2}\,\frac{(x^2 + 1)^3}{3} + C$$

$$\frac{1}{2}\int u^2 \, du = \frac{1}{2}\,\frac{u^{2+1}}{2+1} + C$$

Example 3. u-Substitution: Reversing the Power Chain Rule

Suppose we want to find the following antiderivative:

$$\int (x^3 + 8)^{\frac{-1}{2}} x^2 \, dx$$

Then we can use the following pattern:

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

Let $u = (x^3 + 8)$, $du = 3x^2 dx$, and then try to match the pattern above:

$$\frac{1}{3} \int (x^3 + 8)^{\frac{-1}{2}} 3x^2 \, dx = \frac{1}{3} \frac{(x^3 + 8)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\frac{1}{3} \int u^{\frac{-1}{2}} \, du = \frac{1}{3} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

Example 4. u-Substitution: Reversing the Exponential Chain Rule

Suppose we want to find the following antiderivative: $\int e^{(x^3 + 8)} x^2 \, dx$

Then we can use the following pattern: $\int e^u \, du = e^u + C$

Let $u = (x^3 + 8)$, $du = 3x^2 \, dx$, and then try to match the pattern above:

$$\frac{1}{3} \int e^{(x^3 + 8)} \, 3x^2 \, dx = \frac{1}{3} e^{(x^3 + 8)} + C$$

$$\frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C$$

Example 5. u-Substitution: Reversing the Natural Log Chain Rule

Suppose we want to find the following antiderivative:

$$\int \frac{x^2\,dx}{(x^3 + 8)}$$

Then we can use the following pattern:

$$\int \frac{du}{u} = \ln(u) + C$$

Let $u = (x^3 + 8)$, $du = 3x^2\,dx$, and then try to match the pattern above:

$$\frac{1}{3}\int \frac{3\,x^2\,dx}{(x^3 + 8)} = \frac{1}{3}\ln(x^3 + 8) + C$$

$$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln(u) + C$$

Example 6. u-Substitution: Reversing the Sine Chain Rule

Suppose we want to find the following antiderivative:

$$\int \sin(x^3 + 8)\, x^2\, dx$$

Then we can use the following pattern:

$$\int \sin(u)\, du = -\cos(u) + C$$

Let $u = (x^3 + 8)$, $du = 3x^2\, dx$, and then try to match the pattern above:

$$\frac{1}{3}\int \sin(x^3 + 8)\, 3\, x^2\, dx = -\frac{1}{3}\cos(x^3 + 8) + C$$

$$\frac{1}{3}\int \sin(u)\, du = -\frac{1}{3}\cos(u) + C$$

Example 7. u-Substitution: Reversing the Cosine Chain Rule

Suppose we want to find the following antiderivative:

$$\int \cos(x^3 + 8)\, x^2\, dx$$

Then we can use the following pattern:

$$\int \cos(u)\, du = \sin(u) + C$$

Let $u = (x^3 + 8)$, $du = 3x^2\, dx$, and then try to match the pattern above:

$$\frac{1}{3}\int \cos(x^3 + 8)\, 3\, x^2\, dx = \frac{1}{3}\sin(x^3 + 8) + C$$

$$\frac{1}{3}\int \cos(u)\, du = \frac{1}{3}\sin(u) + C$$

Some u-Substitution Antiderivative Rules

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

$$\int \sin(u) \, du = -\cos(u) + C$$

$$\int e^u \, du = e^u + C$$

$$\int \cos(u) \, du = \sin(u) + C$$

$$\int \frac{du}{u} = \ln(u) + C$$