Polar Coordinates

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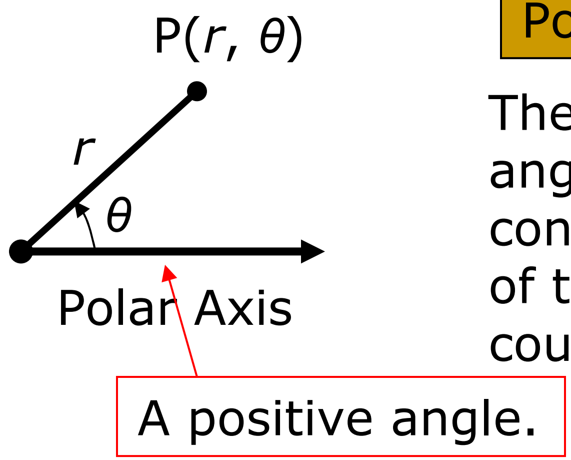
Polar Coordinates

To define the Polar Coordinates of a plane we need first to fix a point which will be called the Pole (or the origin) and a half-line starting from the pole. This half-line is called the Polar Axis.



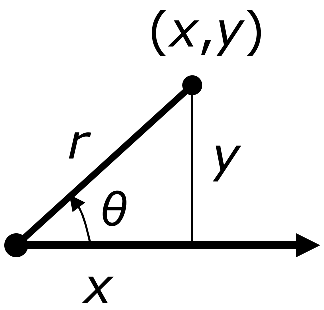
Polar Angles

The Polar Angle θ of a point P, P ≠ pole, is the angle between the Polar Axis and the line connecting the point P to the pole. Positive values of the angle indicate angles measured in the counterclockwise direction from the Polar Axis.

Polar Coordinates

The Polar Coordinates (r,θ) of the point P, P ≠ pole, consist of the distance r of the point P from the Pole and of the Polar Angle θ of the point P. Every (0, θ) represents the pole.

Polar and Cartesian Coordinates



From the right angle triangle in the picture one immediately gets the following correspondence between the Cartesian Coordinates (x,y) and the Polar Coordinates (r,θ) assuming the Pole of the Polar Coordinates is the Origin of the Cartesian Coordinates and the Polar Axis is the positive x-axis.

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$$r^2 = x^2 + y^2$$
$$\tan(\theta) = y/x$$

Using these equations one can easily switch between the Cartesian and the Polar Coordinates.

The Equation of a Line

The Cartesian equation of a general line is of the form

$$ax + by + c = 0$$

If the line in question passes through the origin, $c = 0$.
In this case the Polar Coordinate Equation for the line is

$$\tan(\theta) = -a/b.$$

If the line does not pass through the origin, one needs to substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to the equation of the line. One gets

$$a\, r\cos(\theta) + b\, r\sin(\theta) = c.$$

This yields $r = -\,c/(a\cos(\theta) + b\sin(\theta)).$

One concludes that the equation of a general line in the Polar Coordinates is rather complicated.

The Equation of a Circle

The Cartesian Equation of a Circle with radius r_0 and with center at the origin is

$$x^2 + y^2 = r_0^2.$$

In the Polar Coordinates this equation becomes remarkably simple:

$$r = r_0.$$

Polar Curves

Definition

A Polar Curve consists of all the points (r,θ) satisfying a given equation

$$F(r,\theta) = 0.$$

Often one can solve r form the equation and represent the polar curve in the form

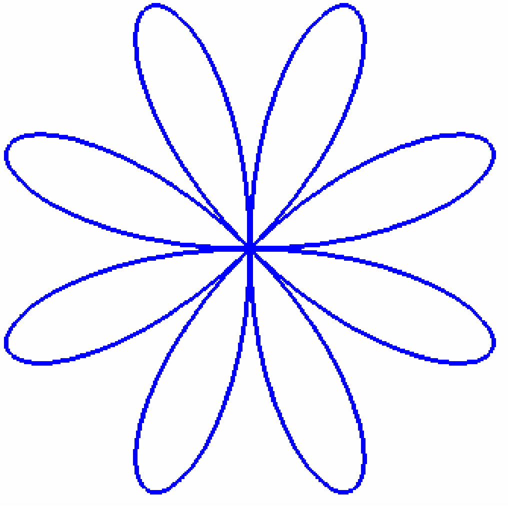
$$r = f(\theta)$$

Plotting Polar Curves by Maple

Use the option "coords = polar" in the plot command to plot curves defined in the polar coordinates. For the Maple command

```
plot([sin(4*t),t,t=0..2*Pi],coords=polar);
```

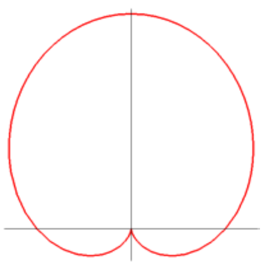
produces the following plot.



Areas of Polar Domains

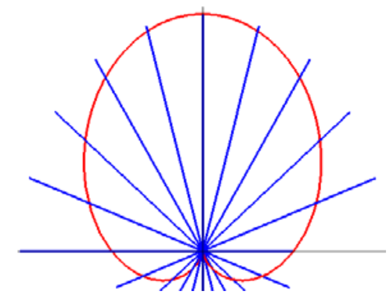
To compute the area of a polar domain, i.e., the area inside the graph of a polar curve, one repeats the same arguments as the ones used in the Cartesian case.

One divides the domain in question by half-line emanating from the origin and forms a Riemann sum approximation for the area of the domain in question.



Polar Curve
$r = 1 + \sin(\theta)$

Divide a polar domain into several sections whose area will be approximated by areas of circular sections.



Approximate the areas of the sections by the areas of sections of circles.

Areas of Polar Domains

Formula

The area enclosed by the graph of the polar curve

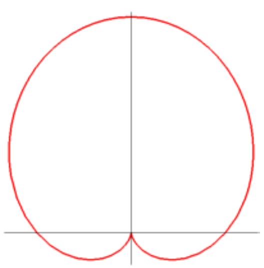
$r = \mathrm{f}(\theta)$

and by the polar lines $\theta = a$ and $\theta = b$ is given by

$$A = \int_a^b \frac{1}{2}\left(\mathrm{f}(\theta)\right)^2 \, d\theta \quad \text{or} \quad A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

Example

The area of the domain enclosed by the Polar Curve $r = 1 + \sin(\theta)$ is computed as follows.



$$A = \frac{1}{2}\int_0^{2\pi} \left(1 + \sin(\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\sin(\theta) + \sin^2(\theta)\right) d\theta = \frac{3\pi}{2}$$