

Indefinite Integral (Antiderivative) Exercises: Tangent Substitution

Max Z.

Indefinite Integrals Exercise 4.1: $\int \frac{1}{x^2 \sqrt{x^2+4}} dx$

$x = 2 \tan \theta$
 $dx = 2 \sec^2 \theta d\theta$

$2\sqrt{4 + \tan^2 \theta} = 2 \sec \theta$

$\int \frac{\sec \theta d\theta}{4 \tan^2 \theta} = \frac{1}{4} \int \frac{\cancel{\cos \theta} d\theta}{\cancel{\cos \theta} \sin^2 \theta}$ $\sin \theta = u$

$\frac{1}{4} \int \frac{du}{u^2} = -\frac{1}{4} \frac{1}{\sin \theta}$

$x^2 = \frac{4 \sin^2 \theta}{1 - \sin^2 \theta} \rightarrow x^2 - x^2 \sin^2 \theta = 4 \sin^2 \theta$

$x^2 = \sin^2 \theta (4 + x^2)$

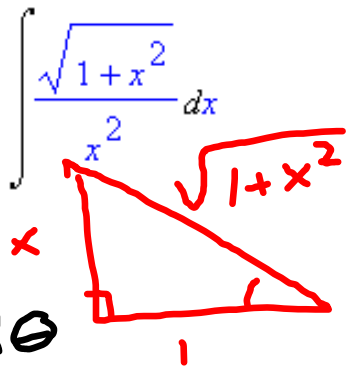
$x = \sin \theta \sqrt{4 + x^2}$

$\frac{1}{\sin \theta} = \frac{\sqrt{4+x^2}}{x}$
 $\int = -\frac{\sqrt{4+x^2}}{4x} + C$



MUSTAFAN

Indefinite Integrals Exercise 4.2:



$$x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

$$\int \frac{\sqrt{1+x^2}}{x^2} dx = \int \frac{\sqrt{1+\tan^2 \theta}}{\tan^2 \theta} \sec^2 \theta d\theta$$

$$\sin \theta = \frac{x}{\sqrt{x^2+1}}$$

$$\cos \theta = \frac{1}{\sqrt{x^2+1}}$$

$$= \int \frac{\sec \theta}{\tan^2 \theta} \cdot \sec^2 \theta d\theta$$

$$= \int \frac{\cos^2 \theta}{\sin^2 \theta} \cdot \frac{1}{\cos^3 \theta} d\theta$$

$$= \int \frac{1}{\sin^2 \theta \cos \theta} d\theta$$

$$= \int \frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \theta \cos \theta}$$

$$= \int \frac{\cos^2 \theta}{\sin^2 \theta \cos \theta} d\theta + \int \frac{\sin^2 \theta}{\sin^2 \theta \cos \theta} d\theta$$

$$= \int \cos \theta \cdot \sin^{-2} \theta d\theta + \int \sec \theta d\theta$$

let $u = \sin \theta$
 $du = \cos \theta d\theta$

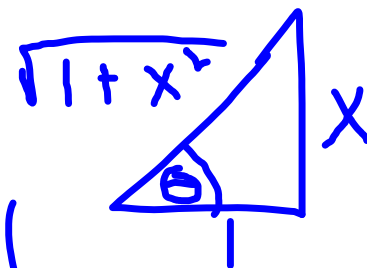
$$= \int u^{-2} du + \ln \left| \frac{\cos x + 1}{|\sin x - 1|} \right|$$

$$= -\frac{x}{\sqrt{x^2+1}} + \ln \left[\frac{|\sqrt{1+x^2}|}{\left| \frac{x}{\sqrt{1+x^2}} - 1 \right|} \right]$$



$$\int \sin^{-2} \theta \cos \theta d\theta + \int \sec \theta d\theta =$$

$$\frac{(\sin \theta)^{-1}}{-1} + \ln |\sec \theta + \tan \theta|$$

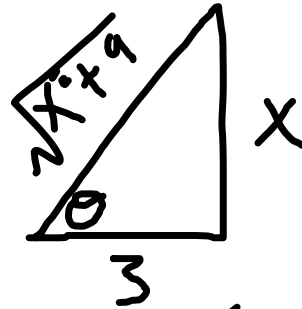


$$= \frac{-\sqrt{1+x^2}}{x} + \ln |\sqrt{1+x^2} + x| + C$$

for

Indefinite Integrals Exercise 4.3:

$$\int \frac{x^3}{\sqrt{x^2+9}} dx$$



$$\frac{1}{3} \int \frac{x^3}{\sqrt{\left(\frac{x}{3}\right)^2 + 1}}$$

$$\tan \theta = \frac{x}{3}$$

$$3 \tan \theta = x$$

$$3 \sec^2 \theta d\theta = dx$$

$$\frac{1}{3} \int \frac{27 \tan^3 \theta}{\sqrt{\tan^2 \theta + 1}} \cdot 3 \sec^2 \theta d\theta$$

$$9 \int \frac{\tan^3 \theta}{\sec \theta} \cdot 3 \sec^2 \theta d\theta$$

$$27 \int \tan^2 \theta \sec \theta d\theta$$

$$27 \int (\sec^2 \theta - 1) \cdot \tan \theta \sec \theta d\theta$$

$$27 \left(\frac{\sec^3 \theta}{3} - \sec \theta \right)$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\sqrt{x^2+9}}{3}$$

$$\frac{(x^2+9)^{3/2}}{3} - 9 \cdot \sqrt{x^2+9} + C$$



Eliz

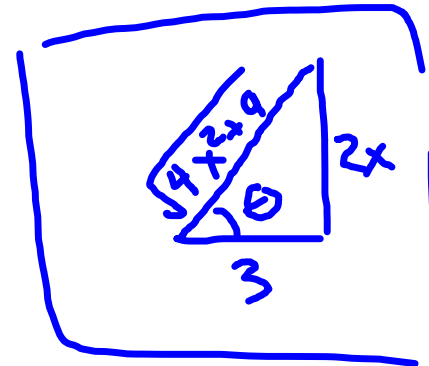
Indefinite Integrals Exercise 4.4:

$$\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}} dx$$

$$\frac{2}{3}x = \tan \theta$$

$$\frac{2}{3}dx = \sec^2 \theta d\theta$$

$$\int \frac{\left(\frac{3}{2}\tan \theta\right)^3}{27(\tan^2 \theta + 1)^{3/2}} \cdot \frac{3}{2} \sec^2 \theta d\theta$$



$$\frac{3}{16} \int \frac{\tan^3 \theta}{\sec \theta} d\theta$$

$$\frac{3}{16} \int \frac{\sin^3 \theta}{\cos^2 \theta} d\theta$$

$$u = \cos \theta$$

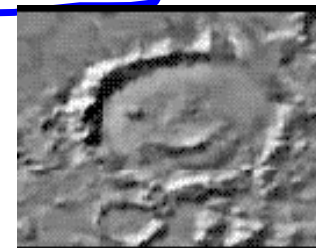
$$du = -\sin \theta d\theta$$

$$\frac{-3}{16} \int \frac{(1-u^2)}{u^2} du$$

$$\frac{3}{16u} + \frac{3u}{16} + C$$

$$\frac{3}{16 \cos \theta} + \frac{3 \cos \theta}{16} + C$$

$$\frac{\sqrt{4x^2+9}}{16} + \frac{9}{16\sqrt{4x^2+9}} + C$$

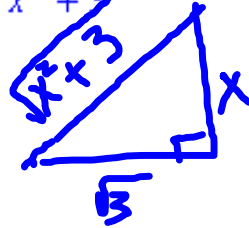


Indefinite Integrals Exercise 4.5:

$$\int \frac{1}{x\sqrt{x^2+3}} dx$$

Amy

$$\tan \theta = \frac{x}{\sqrt{3}}$$



$$\sqrt{3} \tan \theta = x$$

$$\sqrt{3} \sec^2 \theta d\theta = dx$$

$$\int \frac{\sqrt{3} \sec^2 \theta}{3 \tan \theta \sqrt{x^2+3}} d\theta$$

$$\frac{\sqrt{3}}{3} \int \frac{\frac{1}{\cos^2 \theta}}{\frac{\sin \theta}{\cos^2 \theta}} d\theta$$

$$\frac{\sqrt{3}}{3} \int \frac{1}{\sin \theta} d\theta$$

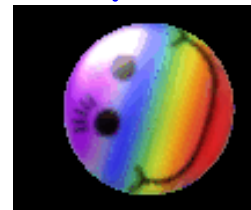
$$\frac{\sqrt{3}}{3} \ln \left| \frac{\sin \theta}{\cos \theta + 1} \right| + C$$

$$\frac{\sqrt{3}}{3} \ln \left| \frac{\frac{x}{\sqrt{x^2+3}}}{\frac{\sqrt{3}}{\sqrt{x^2+3}} + 1} \right| + C$$

$$\frac{\sqrt{3}}{3} \ln \left| \frac{\frac{x}{\sqrt{x^2+3}}}{\frac{\sqrt{3} + \sqrt{x^2+3}}{\sqrt{x^2+3}}} \right| + C$$

$$\frac{\sqrt{3}}{3} \ln \left| \frac{x}{\sqrt{x^2+3}} \cdot \frac{\sqrt{x^2+3}}{\sqrt{3} + \sqrt{x^2+3}} \right| + C$$

$$\frac{\sqrt{3}}{3} \ln \left| \frac{x}{\sqrt{3} + \sqrt{x^2+3}} \right| + C$$



Done A.

Indefinite Integrals Exercise 4.6:

$$\int \frac{x}{(x^2+4)^{\frac{5}{2}}} dx$$

$$\frac{x}{2} = \tan \theta$$

$$x = 2 \tan \theta$$

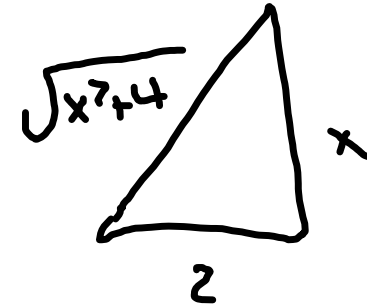
$$dx = 2 \sec^2 \theta d\theta$$

$$\cos \theta = \frac{2}{\sqrt{x^2+4}}$$

$$d\theta = -\frac{du}{\sin \theta}$$

$$\int \frac{x}{(\sqrt{x^2+4})^5}$$

$$-\frac{1}{24} \cos^3 \theta$$



$$\int \frac{2 \tan \theta}{\sqrt{4 \tan^2 \theta + 4}} \cdot 2 \sec^2 \theta d\theta$$

$$-\frac{1}{24} \left(\frac{2}{\sqrt{x^2+4}} \right)^3$$

Actually don't need trig sub here.

$$4 \int \frac{\tan \theta}{22 \sec \theta} \cdot \sec^2 \theta d\theta$$

$$-\frac{1}{3} (x^2+4)^{3/2} + C$$

$$\frac{1}{2} \int \frac{\tan \theta}{\sec \theta} \cdot \sec^2 \theta d\theta$$

$$-\frac{5}{2} (x^2+4)^{3/2} + C$$

$$-\frac{1}{2} \int u^2 du$$

$$-\frac{1}{2} \cdot \frac{u^3}{3}$$



Alex S.

Indefinite Integrals Exercise 4.7:

$$\int \frac{1}{(x^2+2x+2)} dx \quad \left\{ \begin{array}{l} 2\cos^2\theta - 1 = \cos 2\theta \\ \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1) \end{array} \right. \int \frac{1}{2} \cos 2\theta + \frac{1}{2} d\theta$$

$$x+1 = \tan\theta$$

$$(x+1)^2 = \tan^2\theta$$

$$dx = \sec^2\theta d\theta$$

$$\int \frac{1}{[(x^2+2x+1)+1]^2} dx$$

$$\int \frac{1}{[(x+1)^2+1]^2} dx$$

$$\int \frac{\sec^2\theta}{(\tan^2\theta+1)^2} d\theta$$

$$\int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$$\int \cos^2\theta d\theta$$

$$\frac{1}{4} \sin 2\theta + \frac{1}{2} \theta$$

$$\frac{1}{2} \sin\theta \cos\theta + \frac{1}{2} \theta$$

$$\frac{1}{2} \left(\frac{x+1}{\sqrt{x^2+2x+2}} \right) \left(\frac{1}{\sqrt{(x+1)^2+1}} \right) \dots$$

$$\dots + \frac{1}{2} \tan^{-1}(x+1) + C$$



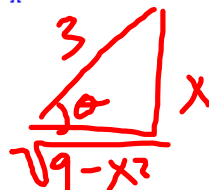
Indefinite Integral (Antiderivative) Exercises: Sine & Secant Substitution

Indefinite Integrals Exercise 4.8:

$$\int \frac{\sqrt{9-x^2}}{x^2} dx$$

$$\begin{aligned} \sin \theta &= \frac{x}{3} \\ 3 \sin \theta &= x \quad dx = 3 \cos \theta d\theta \\ 9 \sin^2 \theta &= x^2 \end{aligned}$$

Timmy



$$\int \frac{\sqrt{9(1-(\frac{x}{3})^2)}}{x^2}$$

$$3 \int \frac{\sqrt{1-\sin^2 \theta} \cdot 3 \cos \theta d\theta}{9 \sin^2 \theta}$$

$$3 \int \frac{\cos \theta}{9 \sin^2 \theta} \cdot 3 \cos \theta d\theta \quad \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \quad \int \frac{\cos \theta}{\sin \theta} \cdot \frac{\cos \theta}{\sin \theta} d\theta$$

4.8 continued

$$\int \frac{\cos \theta \cdot \cos \theta}{\sin \theta \cdot \sin \theta} d\theta$$

$$\int \frac{1 - \sin^2 \theta}{\sin^2 \theta}$$

$$\int \frac{1}{\sin^2 \theta} - \frac{\sin^2 \theta}{\sin^2 \theta} d\theta$$

$$\int \csc^2 \theta - 1 d\theta$$

$$\int \csc^2 \theta - \int d\theta$$

$$-\cot \theta - \theta$$

$$-\left(\frac{\sqrt{9-x^2}}{x}\right) - \sin^{-1}\left(\frac{x}{3}\right)$$



Indefinite Integrals Exercise 4.9:

$$\int \frac{1}{\sqrt{x^2 - 4}} dx$$

David

$$= \int \frac{1}{\sqrt{2\left(\frac{x}{2}\right)^2 - 1}} dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 - 1}} dx$$

$$\frac{x}{2} = \sec \theta$$

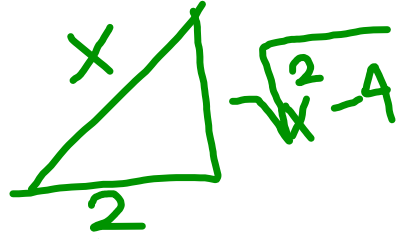
$$dx = 2 \sec \theta \tan \theta d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{\sec^2 \theta - 1}} \cdot 2 \sec \theta \tan \theta d\theta$$

$$= \frac{1}{2} \int \frac{1}{\tan^2 \theta} \cdot 2 \sec \theta \tan \theta d\theta$$

$$= \int \frac{1}{\tan \theta} \cdot \sec \theta \tan \theta d\theta$$

$$= \int \sec \theta d\theta$$



$$= \ln |\sec \theta + \tan \theta| + C$$

$$= \ln \left| \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2} \right| + C$$

$$= \ln \left| \frac{x + \sqrt{x^2 - 4}}{2} \right| + C$$

$$= \ln |x + \sqrt{x^2 - 4}| + C$$

the 2 goes away!

Solver D.

Indefinite Integrals Exercise 4.10:

$$\int \frac{x}{\sqrt{3-2x-x^2}} dx$$

$$-x^2 - 2x + 3 = -(x^2 + 2x - 3)$$
$$-(x^2 + 2x + 1 - 4)$$

$$\frac{1}{2} \int \frac{x}{\sqrt{4-(x+1)^2}} dx$$
$$\frac{1}{2} \int \frac{2 \sin \theta - 1}{\sqrt{1 - \sin^2 \theta}} 2 \cos \theta d\theta$$
$$\frac{1}{2} \int \frac{(2 \sin \theta - 1) 2 \cos \theta d\theta}{\cos \theta}$$

$$\frac{1}{4} (x+1)^2 = \sin^2 \theta$$

$$(x+1)^2 = 4 \sin^2 \theta$$

$$x+1 = 2 \sin \theta$$

$$x = 2 \sin \theta - 1$$

$$dx = 2 \cos \theta d\theta$$

$$\frac{1}{2} \int 4 \sin \theta - 2 - 2 \cos \theta - \theta - 2 \cos \left(\sin^{-1} \left(\frac{x+1}{2} \right) \right) - \sin^{-1} \left(\frac{x+1}{2} \right) - \sqrt{4-(x+1)^2} - \sin^{-1} \left(\frac{x+1}{2} \right) + C$$



Holla

Indefinite Integrals Exercise 4.11:

$$\int x^3 \sqrt{16-x^2} dx$$

$$\int x^3 \sqrt{16(1-(\frac{x}{4})^2)}$$

$$\frac{x}{4} = \sin \theta$$

$$x = 4 \sin \theta$$

$$4 \int x^3 \sqrt{1-(\frac{x}{4})^2}$$

$$dx = 4 \cos \theta d\theta$$

$$x^3 = 64 \sin^3 \theta$$

$$4 \int 64 \sin^3 \theta \sqrt{1-\sin^2 \theta} 4 \cos \theta d\theta$$

$$1024 \int \sin^3 \theta - \sin^5 \theta$$

$$1024 \int (1-\cos^2 \theta) \sin \theta - 1024 \int \sin^5 \theta$$

$$1024 \int \sin \theta - 1024 \int \sin \theta \cos^2 \theta - 1024 \int \sin^5 \theta$$

$$-1024 \cos \theta + 1024 \int -u^2 - 1024 \int \sin \theta - 1024 \int -2 \cos^2 \theta \sin \theta - 1024 \int \sin \theta \cos^4 \theta$$

$$\frac{-1024 \cos^3 \theta}{3} - \frac{1024 \cos^5 \theta}{3}$$

$$-\frac{16}{3} (16-x^2)^{3/2} + \frac{1}{5} (16-x^2)^{5/2}$$



Indefinite Integrals Exercise 4.12:

$$\int \frac{1}{x^2 \sqrt{25x^2 - 9}} dx$$

$$dx = \frac{3}{5} \sec \theta \tan \theta d\theta$$

$$3\sqrt{\sec^2 \theta} = 3 \tan \theta$$

$$x = \frac{3}{5} \sec \theta$$

$$5x = 3 \sec \theta$$

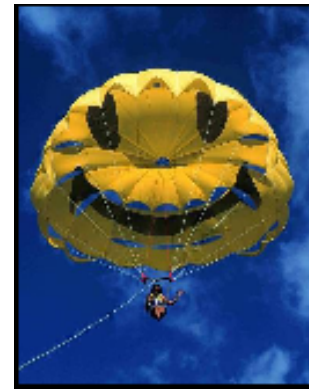
$$\int = \frac{3}{5} \frac{1}{3} \frac{25}{9} \int \frac{\sec \theta \tan \theta d\theta}{\sec^2 \theta \tan \theta} = \frac{1}{\sec \theta} = \cos$$

$$\frac{5}{9} \sin \theta = \frac{3}{5} \frac{1}{\sqrt{1 - \sin^2 \theta}} = x$$

$$\sin^2 \theta = 1 - \frac{9}{25x^2} = \frac{25x^2 - 9}{25x^2}$$

$$\sin \theta = \frac{\sqrt{25x^2 - 9}}{5x}$$

$$\int = \frac{\sqrt{25x^2 - 9}}{5x} \cdot \frac{5}{9} = \frac{\sqrt{25x^2 - 9}}{9x}$$



Elrabt

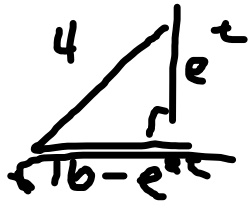
Indefinite Integrals Exercise 4.13:

$$\int e^t \sqrt{16 - e^{2t}} dt$$

$$\int e^t 4 \sqrt{1 - \left(\frac{e^t}{4}\right)^2} dt$$

$\frac{e^t}{4} = \sin \theta$
 $e^t dt = 4 \cos \theta d\theta$
 $dt = \cot \theta d\theta$

$$4 \int 4 \sin \theta \sqrt{1 - \sin^2 \theta} \cdot \cot \theta d\theta$$
$$16 \int \frac{\sin \theta \cdot \cos \theta \cdot \cos \theta d\theta}{\sin \theta}$$



$$16 \int \cos^2 \theta d\theta$$
$$= 8 \sin \theta \cos \theta + 8\theta + C$$
$$= \frac{de^t}{4} \cdot \frac{\sqrt{16 - e^{2t}}}{4} + 8 \sin^{-1} \left(\frac{e^t}{4} \right) + C$$



$$= \frac{e^t \sqrt{16 - e^{2t}}}{2} + 8 \sin^{-1} \left(\frac{e^t}{4} \right) + C$$

Indefinite Integrals Exercise 4.14:

$$\int u^2 \sqrt{a^2 - u^2} du$$

David G.

$$\frac{u}{a} = \sin \theta$$

$$\frac{du}{a} = \cos \theta d\theta$$

$$u^2 = a^2 \sin^2 \theta$$

$$\cos \theta = \frac{\sqrt{a^2 - u^2}}{a}$$

$$a \int u^2 \cos \theta \cdot \cos \theta d\theta$$

$$\int a^3 \sin^2 \theta \cos^2 \theta d\theta$$

$$a^3 \int \frac{\sin^2 2\theta}{4} d\theta$$

$$\frac{a^3}{4} \left(\frac{\theta}{2} - \frac{\sin \theta \cos \theta}{2} \right)$$

$$\frac{a^3}{8} \left(a \sin \frac{u}{a} - \frac{u \sqrt{a^2 - u^2}}{a^2} \right) + C$$



$$94 \int \sin^2 \theta \cos^2 \theta d\theta =$$

$$94 \int (\sin \theta \cos \theta)^2 d\theta =$$

$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$

$$\frac{94}{4} \int (2 \sin \theta \cos \theta)^2 d\theta$$

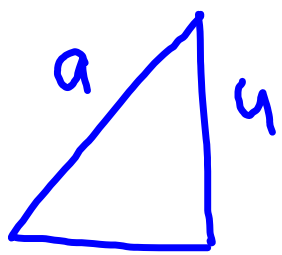
$$\frac{94}{4} \int (\sin 2\theta)^2 d\theta =$$
$$\frac{94}{4} \int \frac{1 - \cos 4\theta}{2} d\theta$$

$$= \frac{a^4}{8} \theta - \frac{a^4}{32} \sin 4\theta$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$



$$\begin{aligned} \sin 4\theta &= 2 \sin 2\theta \cos 2\theta \\ &= 2 \cdot 2 \sin \theta \cos \theta (1 - 2 \sin^2 \theta) \\ &= 4 \frac{y}{a} \cdot \frac{\sqrt{a^2 - y^2}}{a} \left(1 - 2 \frac{y^2}{a^2} \right) \end{aligned}$$



$\sqrt{a^2 - y^2}$

$$= \frac{a^4}{8} \sin^{-1} \left(\frac{y}{a} \right) - \frac{a^4}{32} \cdot \frac{y}{a} \cdot \frac{\sqrt{a^2 - y^2}}{a} \left(1 - 2 \frac{y^2}{a^2} \right)$$

$$= \frac{a^4}{8} \sin^{-1}\left(\frac{u}{a}\right) - a^2 u \sqrt{a^2 - u^2} \left(1 - \frac{2u^2}{a^2}\right)$$

$$= \frac{a^4}{8} \sin^{-1}\left(\frac{u}{a}\right) - u \sqrt{a^2 - u^2} (a^2 - 2u^2)$$

Indefinite Integrals Exercise 4.15: $\int \sqrt{u^2 - a^2} du$

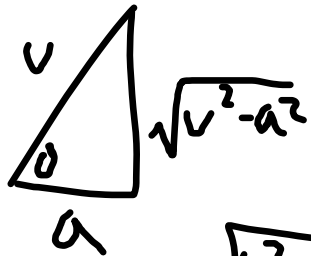
Jon

$$\sec \theta = \frac{u}{a}$$

$$a \sec \theta = u$$

$$a \tan \theta \sec \theta d\theta = du$$

SohCantoa



$$\tan \theta = \frac{\sqrt{u^2 - a^2}}{a}$$

$$\sec \theta = \frac{u}{a}$$

$$a \int \sqrt{\left(\frac{u}{a}\right)^2 - 1} du$$

$$a \int \sqrt{\sec^2 \theta - 1} a \sec \theta \tan \theta d\theta$$

$$a^2 \int \tan^2 \theta \sec \theta d\theta$$

$$a^2 \int (\sec^2 \theta - 1) \sec \theta d\theta$$

$$a^2 \int \sec^3 \theta - \sec \theta d\theta$$

$$a^2 \left(\frac{1}{2} \tan \theta \sec \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| \right)$$

$$\frac{a^2}{2} \tan \theta \sec \theta - \frac{a^2}{2} \ln |\sec \theta + \tan \theta|$$

$$\frac{a^2}{2} \cdot \frac{\sqrt{u^2 - a^2}}{a} \cdot \frac{u}{a} - \frac{a^2}{2} \ln \left| \frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a} \right|$$

$$\frac{u \sqrt{u^2 - a^2}}{2} - \frac{a^2}{2} \ln \left| \frac{u + \sqrt{u^2 - a^2}}{a} \right| + C$$

