Indefinite Integral (Antiderivative) Exercises: Tangent Substitution

Max Z.

Indefinite Integrals Exercise 4.1: $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$2\sqrt{1+\tan^2\theta} = 2\sec\theta$

$\int \frac{\sec\theta\, d\theta}{4\tan^2\theta} = \frac{1}{4}\int \frac{\cos^{\cancel{2}}\theta\, d\theta}{\cancel{\cos\theta}\,\sin^2\theta}$ $\sin\theta = u$

$\frac{1}{4}\int \frac{du}{u^2} = \frac{-1}{4}\,\frac{1}{\sin\theta}$

$x^2 = \frac{4\sin^2\theta}{1-\sin^2\theta} \longrightarrow x^2 - x^2\sin^2\theta = 4\sin^2\theta$

$x^2 = \sin^2\theta\,(4+x^2)$

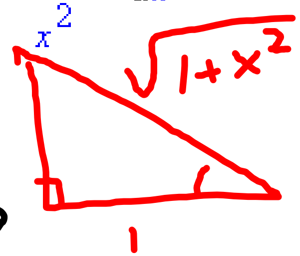
$x = \sin\theta\sqrt{4+x^2}$

$\frac{1}{\sin\theta} = \frac{\sqrt{4+x^2}}{x}$

$\int = \frac{-\sqrt{4+x^2}}{4x} + C$

Mustafa

Indefinite Integrals Exercise 4.2: $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\,dx$



$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta}\sec^2\theta\, d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\cdot \sec^2\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos^3\theta}\, d\theta$$

$$= \int \frac{1}{\sin^2\theta\cos\theta}\, d\theta$$

$$= \int \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos\theta}$$

$\sin\theta = \dfrac{x}{\sqrt{x^2+1}}$

$\cos\theta = \dfrac{1}{\sqrt{x^2+1}}$

$$= \int \frac{\cos^2\theta}{\sin^2\theta\cos\theta}\, d\theta + \int \frac{\sin^2\theta}{\sin^2\theta\cos\theta}\, d\theta$$

$$= \int \cos\theta\cdot\sin^{-2}\theta\, d\theta + \int \sec\theta\, d\theta$$

let $u = \sin\theta$

$du = \cos\theta\, d\theta$

?

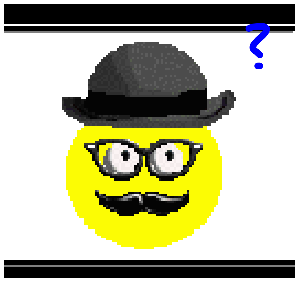
$$= \int u^{-2}\,du + \ln\left[\frac{|\cos x|}{|\sin x - 1|}\right]$$

?

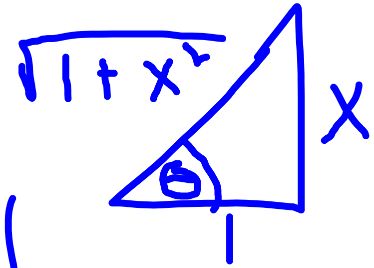
$$-\frac{x}{\sqrt{x^2+1}} + \ln\left[\frac{\left|\frac{1}{\sqrt{1+x^2}}\right|}{\left|\frac{x}{\sqrt{1+x^2}} - 1\right|}\right]$$

?

?



$$\int \sin^{-2}\theta\cos\theta\,d\theta + \int \sec\theta\,d\theta =$$

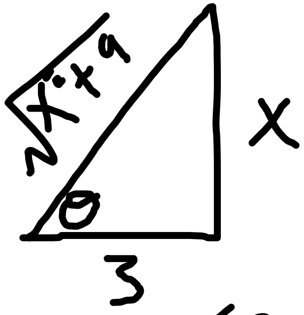


$$\frac{(\sin\theta)^{-1}}{-1} + \ln|\sec\theta + \tan\theta|$$

$$= -\frac{\sqrt{1+x^2}}{x} + \ln\left|\sqrt{1+x^2} + x\right| + C$$

for

Indefinite Integrals Exercise 4.3: $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx$



$$\frac{1}{3}\cdot\int\frac{x^3}{\sqrt{\left(\frac{x}{3}\right)^2+1}}$$

$\tan\theta = \frac{x}{3}$

$3\tan\theta = x$

$3\sec^2\theta\, d\theta = dx$

$$\frac{1}{3}\cdot\int\frac{27\tan^3\theta}{\sqrt{\tan^2\theta+1}}\cdot 3\sec^2\theta\, d\theta$$

$$9\int\frac{\tan^3\theta}{\cancel{\sec\theta}}\cdot 3\sec^{\cancel{2}}\theta\, d\theta$$

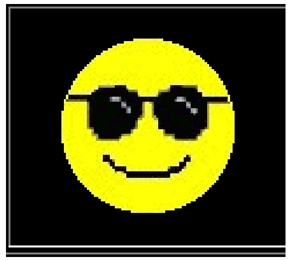
$$27\int\tan^3\theta\sec\theta\, d\theta$$

$$27\int(\sec^2\theta-1)\cdot\tan\theta\sec\theta\, d\theta$$

$$27\cdot\left(\frac{\sec^3\theta}{3}-\sec\theta\right)$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{x^2+9}}{3}$$

$$\boxed{\frac{(x^2+9)^{3/2}}{3} - 9\cdot\sqrt{x^2+9}+C}$$



Eliz

Indefinite Integrals Exercise 4.4: $\displaystyle\int \frac{x^3}{(4x^2+9)^{\left(\frac{3}{2}\right)}}\,dx$

$$\frac{2}{3}x = \tan\theta$$

$$\frac{2}{3}dx = \sec^2\theta\, d\theta$$

$$\int \frac{\left(\frac{3}{2}\tan\theta\right)^3}{27(\tan^2\theta+1)^{3/2}} \cdot \frac{3}{2}\sec^2\theta\, d\theta$$

$$\frac{3}{16}\int \frac{\tan^3\theta}{\sec\theta}\, d\theta$$

$$\frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$u = \cos\theta$$

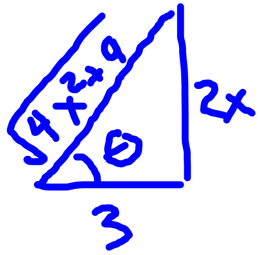
$$du = -\sin\theta\, d\theta$$

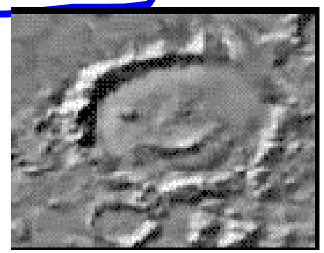
$$\frac{-3}{16}\int \frac{(1-u^2)}{u^2}\, du$$

$$\frac{3}{16u} + \frac{3u}{16} + C$$

$$\frac{3}{16\cos\theta} + \frac{3\cos\theta}{16} + C$$

$$\boxed{\frac{\sqrt{4x^2+9}}{16} + \frac{9}{16\sqrt{4x^2+9}} + C}$$

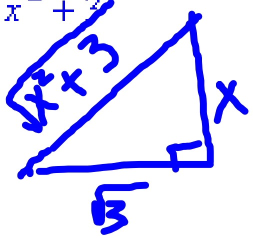




Indefinite Integrals Exercise 4.5: $\displaystyle\int \frac{1}{x\sqrt{x^2+3}}\,dx$

Amy

$\tan\theta = \frac{x}{\sqrt{3}}$



$\sqrt{3}\tan\theta = x$

$\sqrt{3}\sec^2\theta\, d\theta = dx$

$\displaystyle\int \frac{\sqrt{3}\sec^2\theta\, d\theta}{3\tan\theta\sqrt{\cancel{\tan^2\theta+1}}^{\,\sec\theta}}$

$\displaystyle\frac{\sqrt{3}}{3}\int \frac{\frac{1}{\cos^2\theta}}{\frac{\sin\theta}{\cos^2\theta}}\,d\theta$

$\displaystyle\frac{\sqrt{3}}{3}\int \frac{1}{\sin\theta}\,d\theta$

$\displaystyle\frac{\sqrt{3}}{3}\ln\left|\frac{\sin\theta}{\cos\theta+1}\right| + C$

$\displaystyle\frac{\sqrt{3}}{3}\ln\left|\frac{\frac{x}{\sqrt{x^2+3}}}{\frac{\sqrt{3}}{\sqrt{x^2+3}}+1}\right| + C$

$\displaystyle\frac{\sqrt{3}}{3}\ln\left|\frac{\frac{x}{\sqrt{x^2+3}}}{\frac{\sqrt{3}+\sqrt{x^2+3}}{\sqrt{x^2+3}}}\right| + C$

$\displaystyle\frac{\sqrt{3}}{3}\ln\left|\frac{x}{\cancel{\sqrt{x^2+3}}}\cdot\frac{\cancel{\sqrt{x^2+3}}}{\sqrt{3}+\sqrt{x^2+3}}\right| + C$

$\displaystyle\frac{\sqrt{3}}{3}\ln\left|\frac{x}{\sqrt{3}+\sqrt{x^2+3}}\right| + C$

Dane A.

Indefinite Integrals Exercise 4.6: $\int \frac{x}{(x^2+4)^{\left(\frac{5}{2}\right)}}\,dx$

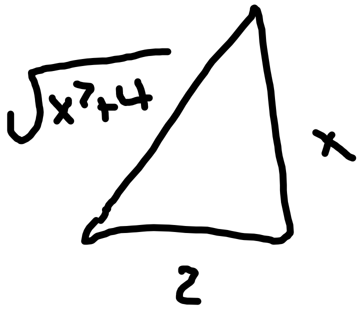
$\frac{x}{2} = \tan\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\cos\theta = u$

$d\theta = \frac{-du}{\sin\theta}$



$\int \frac{x}{\left(\sqrt{x^2+4}\right)^5}$

$\int \frac{2\tan\theta}{\left(\sqrt{4\tan^2\theta+4}\right)^5} \cdot 2\sec^2\theta\, d\theta$

$4\int \frac{\tan\theta}{32\sec^5\theta} \cdot \sec^2\theta\, d\theta$

$\frac{1}{8}\int \frac{\tan\theta}{\sec^3\theta} \cdot \cancel{\sec^2\theta}\, d\theta$

$-\frac{1}{8}\int u^2\, du$

$-\frac{1}{24} \cdot u^3$

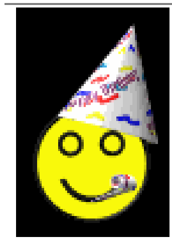
$-\frac{1}{24}\cos^3\theta$

$-\frac{1}{24}\left(\frac{2}{\sqrt{x^2+4}}\right)^3$

$\boxed{-\frac{1}{3(x^2+4)^{3/2}} + C}$

Actually don't need trig sub here.

$\frac{1}{2}\int (x^2+4)^{-\frac{5}{2}}\, 2x\, dx$



Alex S.

Indefinite Integrals Exercise 4.7: $\int \frac{1}{(x^2+2x+2)^2}\,dx$

$2\cos^2\theta - 1 = \cos 2\theta$

$\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$

$$\int \frac{1}{2}\cos 2\theta + \frac{1}{2}\,d\theta$$

$x+1 = \tan\theta$

$(x+1)^2 = \tan^2\theta$

$dx = \sec^2\theta\,d\theta$

$$\int \frac{1}{[(x^2+2x+1-1)+2]^2}\,dx$$

$$\int \frac{1}{[(x+1)^2+1]^2}\,dx$$

$$\int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta$$

$$\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

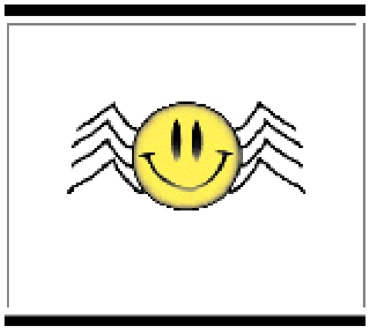
$$\int \cos^2\theta\,d\theta$$

$\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta$

$\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta$

$\frac{1}{2}\left(\frac{x+1}{\sqrt{(x+1)^2+1}}\right)\left(\frac{1}{\sqrt{(x+1)^2+1}}\right)\ldots$

$\ldots + \frac{1}{2}\tan^{-1}(x+1) + C$



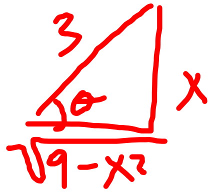
Indefinite Integral (Antiderivative) Exercises: Sine & Secant Substitution

Indefinite Integrals Exercise 4.8: $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$\sin\theta = \frac{x}{3}$

$3\sin\theta = x \quad dx = 3\cos\theta\, d\theta$

$9\sin^2\theta = x^2$



Timmy

$$\int \frac{\sqrt{9\left(1-\left(\frac{x}{3}\right)^2\right)}}{x^2}$$

$$3\int \frac{\sqrt{1-\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$$3\int \frac{\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta \qquad \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \qquad \int \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta$$

4.8 continued

$$\int \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$\int \frac{1-\sin^2\theta}{\sin^2\theta}$$

$$\int \frac{1}{\sin^2\theta} - \frac{\sin^2\theta}{\sin^2\theta}\, d\theta$$

$$\int \csc^2\theta - 1 \; d\theta$$

$$\int \csc^2\theta - \int d\theta$$

$$-\cot\theta - \theta$$

$$-\left(\frac{\sqrt{9-x^2}}{x}\right) - \sin^{-1}\left(\frac{x}{3}\right)$$

Indefinite Integrals Exercise 4.9: $\int \frac{1}{\sqrt{x^2-4}}\,dx$

David L.: $= \int \frac{1}{2\sqrt{(\frac{x}{2})^2-1}}$

$= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{x}{2})^2-1}}\,dx$

$\frac{x}{2} = \sec\theta$

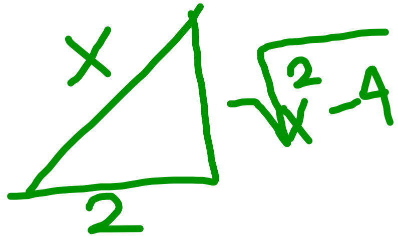
$dx = 2\sec\theta\tan\theta\,d\theta$

$= \frac{1}{2}\int \frac{1}{\sqrt{\sec^2\theta-1}} \cdot 2\sec\theta\tan\theta\,d\theta$

$= \frac{1}{2}\int \frac{1}{\sqrt{\tan^2\theta}} \cdot 2\sec\theta\tan\theta\,d\theta$

$= \int \frac{1}{\cancel{\tan\theta}} \cdot \sec\theta\cancel{\tan\theta}\,d\theta$

$= \int \sec\theta\,d\theta$



$= \ln|\sec\theta+\tan\theta|+C$

$= \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|+C$

$= \ln\left|\frac{x+\sqrt{x^2-4}}{2}\right|+C$

$= \ln\left|x+\sqrt{x^2-4}\right|+C$

the 2 goes away!

Saner D.

Indefinite Integrals Exercise 4.10: $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

$-x^2-2x+3 = -(x^2+2x-3)$

$-(x^2+2x+1-4)$

$\sqrt{4-(x+1)^2}$

$\frac{1}{2}\int \frac{x}{\sqrt{1-\frac{1}{4}(x+1)^2}}$

$\frac{1}{2}\int \frac{2\sin\theta-1}{\sqrt{1-\sin^2\theta}}\,2\cos\theta\,d\theta$

$\frac{1}{2}\int \frac{(2\sin\theta-1)\,2\cos\theta\,d\theta}{\cos\theta}$

$\frac{1}{4}(x+1)^2 = \sin^2\theta$

$(x+1)^2 = 4\sin^2\theta$

$x+1 = 2\sin\theta$

$x = 2\sin\theta-1$

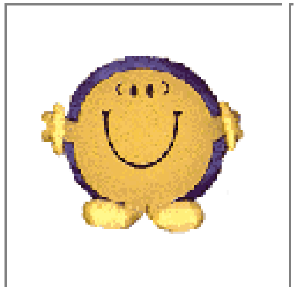
$dx = 2\cos\theta\,d\theta$

$\frac{1}{2}\int 4\sin\theta-2$

$-2\cos\theta-\theta$

$-2\cos\left(\sin^{-1}\left(\frac{x+1}{2}\right)\right)-\sin^{-1}\frac{x+1}{2}$

$-\sqrt{4-(x+1)^2}-\sin^{-1}\left(\frac{x+1}{2}\right)+C$



HollA

Indefinite Integrals Exercise 4.11: $\int x^3\sqrt{16-x^2}\,dx$

$$\boxed{-\frac{16}{3}\left(16-x^2\right)^{\frac{3}{2}}+\frac{1}{5}\left(16-x^2\right)^{\frac{5}{2}}}$$

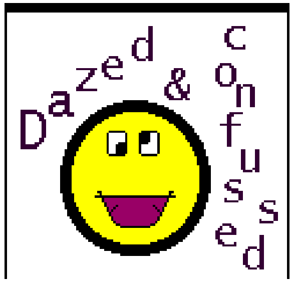
$\int x^3\sqrt{16\left(1-\left(\frac{x}{4}\right)^2\right)}$ $\quad \frac{x}{4}=\sin\theta$

$4\int x^3\sqrt{1-\left(\frac{x}{4}\right)^2}$ $\quad x=4\sin\theta$

$dx=4\cos\theta\,d\theta$

$x^3=64\sin^3\theta$

$4\int 64\sin^3\theta\sqrt{1-\sin^2\theta}\,4\cos\theta\,d\theta$



$1024\int\sin^3\theta-\sin^5\theta$

$1024\int(1-\cos^2\theta)\sin\theta-1024\int\sin^5\theta$

$1024\int\sin\theta-1024\int\sin\theta\cos^2\theta-1024\int\sin^5\theta$

$-1024\cos\theta \quad +1024\int-u^2-1024\int\sin\theta-1024\int-2\cos^2\theta\sin\theta-1024\int\sin\theta\cos^4$

$$\frac{-1024\cos^3\theta}{3}-\frac{1024\cos^5\theta}{3}$$

Indefinite Integrals Exercise 4.12: $\displaystyle\int \frac{1}{x^2\sqrt{25x^2-9}}\,dx$

$dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta$

$x = \frac{3}{5}\sec\theta$

$5x = 3\sec\theta$

$3\sqrt{\sec^2\theta} = 3\tan\theta$

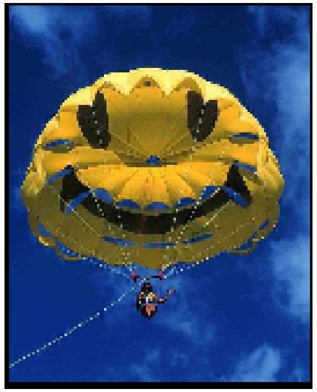
$\int = \frac{\cancel{3}}{5}\cdot\frac{1}{\cancel{3}}\cdot\frac{25}{9}\int \frac{\sec\theta\,\cancel{\tan\theta}\,d\theta}{\sec^2\theta\,\cancel{\tan\theta}} = \frac{1}{\sec\theta} = \cos$

$\frac{5}{9}\sin\theta = \frac{3}{5}\frac{1}{\sqrt{1-\sin^2\theta}} = x$

$\sin^2\theta = 1 - \frac{9}{25x^2} \cdot \frac{25x^2-9}{25x^2}$

$\sin\theta = \frac{\sqrt{25x^2-9}}{5x}$

$\int = \frac{\sqrt{25x^2-9}}{5x}\cdot\frac{5}{9} = \frac{\sqrt{25x^2-9}}{9x}$



Elizabeth

Indefinite Integrals Exercise 4.13: $\int e^{t}\sqrt{16-e^{(2t)}}\,dt$

$$\int e^{t}\,4\sqrt{1-\left(\frac{e^{t}}{4}\right)^{2}}\,dt$$

$$\frac{e^{t}}{4} = \sin\theta$$

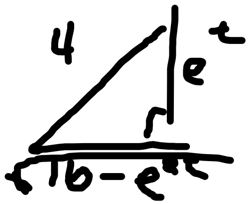
$$e^{t}dt = 4\cos\theta\, d\theta$$

$$dt = \cot\theta\, d\theta$$

$$4\int 4\sin\theta\sqrt{1-\sin^{2}\theta}\cdot\cot d\theta$$

$$16\int\frac{\sin\theta\cdot\cos\theta\cdot\cos\theta\, d\theta}{\sin\theta}$$

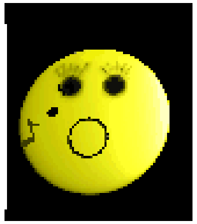
$$16\int\cos^{2}\theta\, d\theta$$



$$= 8\sin\theta\cos\theta + 8\theta + C$$

$$= 8\frac{e^{t}}{4}\cdot\frac{\sqrt{16-\cancel{2}e^{2t}}}{4} + 8\sin^{-1}\left(\frac{e^{t}}{4}\right) + C$$

$$= \frac{e^{t}\sqrt{16-\cancel{2}e^{2t}}}{2} + 8\sin^{-1}\left(\frac{e^{t}}{4}\right) + C$$



Indefinite Integrals Exercise 4.14: $\int u^2 \sqrt{a^2 - u^2}\, du$

David G.

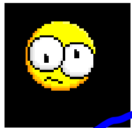
$$\frac{u}{a} = \sin\theta$$

$$\frac{du}{a} = \cos\theta\, d\theta$$

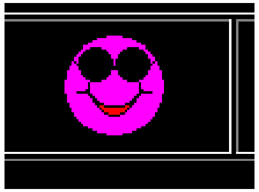
$$u^2 = a^2 \sin^2\theta$$

$$\cos\theta = \frac{\sqrt{a^2 - u^2}}{a}$$

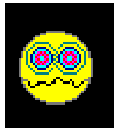
$$a\int u^2 \cos\theta \cdot \cos\theta\, d\theta$$



$$\int a^3 \sin^2\theta \cos^2\theta\, d\theta$$



$$a^3 \int \frac{\sin^2 2\theta}{4}\, d\theta$$



$$\frac{a^3}{4}\left(\frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}\right)$$

$$\frac{a^3}{8}\left(\mathrm{asin}\frac{u}{a} - \frac{u\sqrt{a^2 - u^2}}{a^2}\right) + C$$

$$a^4\int \sin^2\theta\cos^2\theta\,d\theta =$$

$$a^4\int (\sin\theta\cos\theta)^2\,d\theta =$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\frac{a^4}{4}\int (2\sin\theta\cos\theta)^2\,d\theta$$

$$\frac{a^4}{4}\int (\sin 2\theta)^2\,d\theta =$$

$$\rightarrow \frac{a^4}{4}\int \frac{1-\cos 4\theta}{2}\,d\theta$$

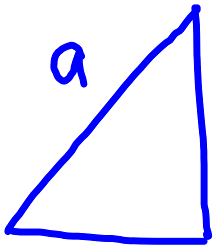
$$= \frac{a^4}{8}\theta - \frac{a^4}{32}\sin 4\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin 4\theta = 2\sin 2\theta \cos 2\theta$$

$$= 2\cdot 2\sin\theta\cos\theta\left(1-2\sin^2\theta\right)$$

$$= 4\,\frac{u}{a}\cdot\frac{\sqrt{a^2-u^2}}{a}\left(1-2\frac{u^2}{a^2}\right)$$



$$= \frac{a^4}{8}\sin^{-1}\left(\frac{u}{a}\right) - \frac{a^{\cancel{4}\,2}}{\cancel{32}\,8}\cdot\frac{\cancel{4}u}{\cancel{a}}\,\frac{\sqrt{a^2-u^2}}{\cancel{a}}\left(1-2\frac{u^2}{a^2}\right)$$

$$= \frac{a^4}{8} \sin^{-1}\left(\frac{u}{a}\right) - a^2 u \sqrt{a^2 - u^2}\left(1 - \frac{2u^2}{a^2}\right)$$

$$= \frac{a^4}{8} \sin^{-1}\left(\frac{u}{a}\right) - u \sqrt{a^2 - u^2}\left(a^2 - 2u^2\right)$$

Indefinite Integrals Exercise 4.15: $\int \sqrt{u^2 - a^2}\, du$

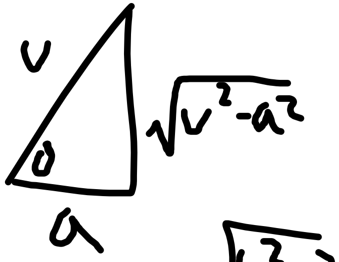
Jon

$\sec\theta = \frac{u}{a}$

$a\sec\theta = u$

$a\tan\theta\sec\theta\, d\theta = du$

sohcahtoa



$\tan\theta = \frac{\sqrt{u^2-a^2}}{a}$

$\sec = \frac{u}{a}$

$$a\int\sqrt{\left(\frac{u}{a}\right)^2-1}\, du$$

$$a\int\sqrt{\sec^2\theta-1}\; a\sec\theta\tan\theta\, d\theta$$

$$a^2\int\tan^2\theta\sec\theta\, d\theta$$

$$a^2\int(\sec^2\theta-1)\sec\theta\, d\theta$$

$$a^2\int\sec^3\theta-\sec\theta\, d\theta$$

$$a^2\left(\frac{1}{2}\tan\theta\sec\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right)$$

$$\frac{a^2}{2}\tan\theta\sec\theta - \frac{a^2}{2}\ln|\sec\theta+\tan\theta|$$

$$\frac{\cancel{a^2}}{2}\cdot\frac{\sqrt{u^2-a^2}}{\cancel{a}}\cdot\frac{u}{\cancel{a}} - \frac{a^2}{2}\ln\left|\frac{u}{a}+\frac{\sqrt{u^2-a^2}}{a}\right|$$

$$\boxed{\frac{u\sqrt{u^2-a^2}}{2} - \frac{a^2}{2}\ln\left|\frac{u+\sqrt{u^2-a^2}}{a}\right| + C}$$