3D Vector Analysis

BC Calculus Fall 2005

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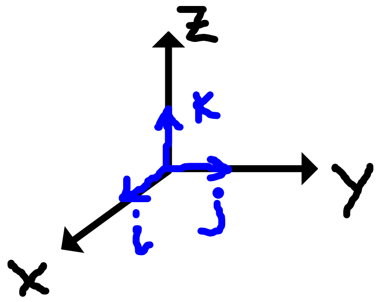
Standard Basis: unit directions

$i = [1,0,0]$

$j = [0,1,0]$

$k = [0,0,1]$



$v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k$

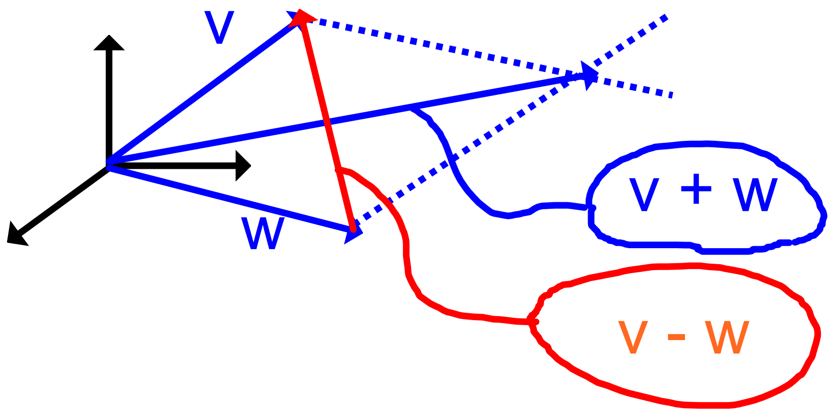
Magnitude or Length of v: $||v|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Direction of v: $\frac{v}{||v||} = \left[\frac{v_1}{||v||}, \frac{v_2}{||v||}, \frac{v_3}{||v||} \right]$

Exercise 1A.
Show that the direction vector above has unit length.

Parallelogram Law for Vector Addition

$v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k \quad w = [w_1, w_2, w_3] = w_1 i + w_2 j + w_3 k$



$v + w = [v_1 + w_1, v_2 + w_2, v_3 + w_3] \quad v - w = [v_1 - w_1, v_2 - w_2, v_3 - w_3]$

Definition 1. Dot or Scalar product of two vectors.

$v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k \quad w = [w_1, w_2, w_3] = w_1 i + w_2 j + w_3 k$

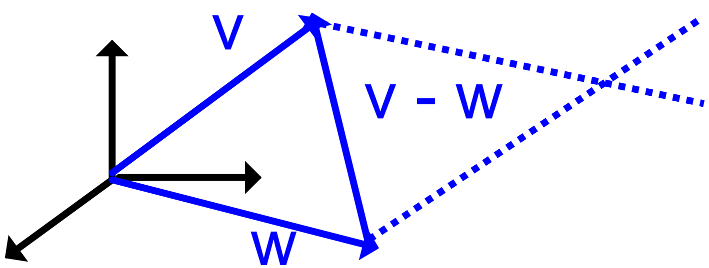
$$v \bullet w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Theorem 1.

1. $v \bullet w = w \bullet v$
2. $av \bullet w = a(v \bullet w) = v \bullet aw$
3. $u \bullet (v + w) = (u \bullet v) + (u \bullet w)$
4. $v \bullet v = ||v||^2$

Exercise 1B. Prove Theorem 1.

Theorem 2. $v \bullet w = ||v|| \, ||w|| \cos\theta$



Exercise 2.
Prove Theorem 2 using the Law of Cosines.

Corollary 1.

1. $\theta = \cos^{-1}\left(\frac{v \bullet w}{\|v\|\|w\|}\right)$

2. $i \bullet i = j \bullet j = k \bullet k = 1$

3. $i \bullet j = j \bullet k = k \bullet i = 0$

Corollary 2. (Perpendicular or Orthogonal vectors)

If v and w are nonzero vectors, then $v \perp w$ if and only if $v \bullet w = 0$.

Exercise 3.
Prove Corollaries 1 and 2.

Review

Determinant of a 2x2 Matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

$$\det(A) = a_{1,1}\, a_{2,2} - a_{1,2}\, a_{2,1}$$

$$A = \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$\det(A) = 6$$

Definition 2. Determinant of a 3x3 Matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$\det(A) = a_{1,1}\, det \begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix} - a_{1,2}\, det \begin{bmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{bmatrix} + a_{1,3}\, det \begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\det(A) = -5$$

Exercise 4.

Compute the determinant of each of the following matrices:

$$\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}, \begin{bmatrix} -5 & 6 \\ -7 & -2 \end{bmatrix}, \begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$$

Answers:

$22, 0, 52, -3\sqrt{6}, 0, -65, -4, -123$

Definition 3. Cross or Scalar product of two vectors.

$v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k \quad w = [w_1, w_2, w_3] = w_1 i + w_2 j + w_3 k$

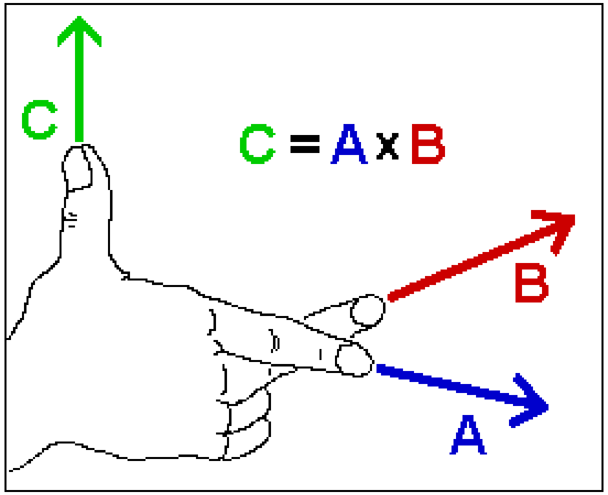
$$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = i(v_2 w_3 - w_2 v_3) - j(v_1 w_3 - w_1 v_3) + k(v_1 w_2 - w_1 v_2)$$

Theorem 3.

1. $v \times w = -(w \times v)$
2. $av \times w = a(v \times w) = v \times aw$
3. $u \times (v + w) = (u \times v) + (u \times w)$
4. $v \times v = 0 = [0,0,0]$
5. $||v \times w|| = ||v||\,||w|| \sin\theta$

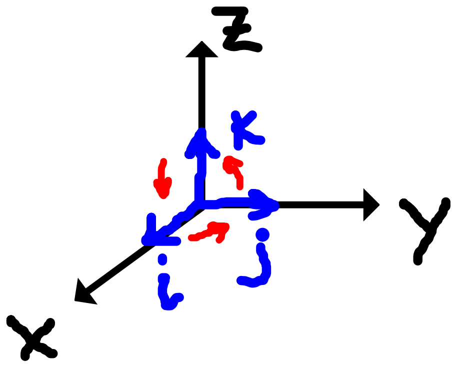
Exercise 5. Prove Theorem 3.

Right Hand Rule



Corollary 3.

1. $i \times i = j \times j = k \times k = 0 = [0,0,0]$
2. $i \times j = k$, $j \times k = i$, $k \times i = j$
3. $||v \times w||$ is the area of the parallelogram made by v and w.
4. If v and w are nonzero vectors, then $v \parallel w$ if and only if $v \times w = 0$.



Exercise 6. Prove Corollary 3.

Exercise 7.

(1) Let v = [0, 1, 2], w = [3, 1, 0], and find $v \times w$ and $w \times v$.

(2) Let u = [-4, 0, 3], v = [2, -1, 0], w = [0, 2, 5], and show

$$u \times (v \times w) \neq (u \times v) \times w.$$

(3) Find two unit vectors orthogonal to both [1, -1, 1], and [0, 4, 4].

(4) Same as (3) for i + j and i - j + k.

(5) Find the area of the parallelogram with vertices P(0,0,0), Q(5,0,0), R(2,6,6), S(7,6,6).

Exercise 8. For each set of three points below find a vector orthogonal to the plane determined by the three points, and find the area of the triangle determined by the three points.

(1) P(1,0,0), Q(0,2,0), R(0,0,3).

(2) P(1,0,-1), Q(2,4,5), R(3,1,7).

(3) P(0,0,0), Q(1,-1,1), R(4,3,7).

(4) P(2,0,-3), Q(3,1,0), R(5,2,2).