Infinite Series

Calculus One and Several Variables, Chapter 13

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3.1 ····SIGMA NOTATION

To indicate the sequence

$$\{1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots\}$$

we can set $a_n = (\frac{1}{2})^{n-1}$ and write

$$\{a_1, a_2, a_3, \ldots\}.$$

We can also set $b_n = (\frac{1}{2})^n$ and write

$$\{b_0, b_1, b_2, \ldots\},$$

thereby beginning with the index 0. More generally, we can set $c_n = (\frac{1}{2})^{n-p}$ and write

$$\{c_p, c_{p+1}, c_{p+2}, \ldots\},$$

thereby beginning with the index p. In this chapter we often begin with an index other than 1.

The symbol Σ is the capital Greek letter "sigma." We write

$$(1) \qquad \sum_{k=0}^{n} a_k$$

(read "the sum of the a sub k from k equals 0 to k equals n") to indicate the sum

$$a_0 + a_1 + \cdots + a_n.$$

More generally, if $n \geq m$, we write

(2) $$\sum_{k=m}^{n} a_k$$

to indicate the sum

$$a_m + a_{m+1} + \cdots + a_n.$$

In (1) and (2) the letter "k" is being used as a "dummy" variable. That is, it can be replaced by any letter not already engaged. For instance,

$$\sum_{i=3}^{7} a_i, \qquad \sum_{j=3}^{7} a_j, \qquad \sum_{k=3}^{7} a_k$$

can all be used to indicate the sum

$$a_3 + a_4 + a_5 + a_6 + a_7.$$

Translating

$$(a_0 + \cdots + a_n) + (b_0 + \cdots + b_n) = (a_0 + b_0) + \cdots + (a_n + b_n),$$
$$\alpha(a_0 + \cdots + a_n) = \alpha a_0 + \cdots + \alpha a_n,$$
$$(a_0 + \cdots + a_m) + (a_{m+1} + \cdots + a_n) = a_0 + \cdots + a_n$$

into the Σ-notation, we have

$$\sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (a_k + b_k), \qquad \alpha \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} \alpha a_k,$$
$$\sum_{k=0}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=0}^{n} a_k.$$

At times it's convenient to change indices. In this connection note that

$$\sum_{k=j}^{n} a_k = \sum_{i=0}^{n-j} a_{i+j}. \qquad \text{(set } i = k - j\text{)}$$

Both expressions are abbreviations for $a_j + a_{j+1} + \cdots + a_n$.

You can familiarize yourself further with this notation by doing the exercises below, but first one more remark. If all the a_k are equal to some fixed number x, then

$$\sum_{k=0}^{n} a_k \quad \text{can be written} \quad \sum_{k=0}^{n} x.$$

Obviously then

$$\sum_{k=0}^{n} x = \overbrace{x + x + \cdots + x}^{n+1} = (n+1)x.$$

In particular

$$\sum_{k=0}^{n} 1 = n + 1.$$

EXERCISES

Evaluate the following expressions.

1. $\sum_{k=0}^{2} (3k+1)$.

2. $\sum_{k=1}^{4} (3k-1)$.

3. $\sum_{k=0}^{3} 2^k$.

4. $\sum_{k=0}^{3} (-1)^k 2^k$.

5. $\sum_{k=0}^{3} (-1)^k 2^{k+1}$.

6. $\sum_{k=2}^{5} (-1)^{k+1} 2^{k-1}$.

7. $\sum_{k=1}^{4} \frac{1}{2^k}$.

8. $\sum_{k=2}^{5} \frac{1}{k!}$.

9. $\sum_{k=3}^{5} \frac{(-1)^k}{k!}$.

10. $\sum_{k=2}^{4} \frac{1}{3^{k-1}}$.

11. $\sum_{k=0}^{3} (\frac{1}{2})^{2k}$.

12. $\sum_{k=1}^{3} (-1)^{k+1} (\frac{1}{2})^{2k-1}$.

Express in sigma notation.

13. the lower sum $m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n$.

14. the upper sum $M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n$.

15. the Riemann sum $f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n$.

16. $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$.

17. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$.

18. $a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n$.

19. $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$.

20. $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$.

21. $1 - 2x + 3x^2 - 4x^3 + 5x^4$.

22. $3x - 4x^2 + 5x^3 - 6x^4$.

Write the following sums as

$$\sum_{k=3}^{10} a_k \quad \text{and} \quad \sum_{i=0}^{7} a_{i+3}:$$

23. $\frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{10}}$.

24. $\frac{3^3}{3!} + \frac{4^4}{4!} + \cdots + \frac{10^{10}}{10!}$.

25. $\frac{3}{4} - \frac{4}{5} + \cdots - \frac{10}{11}$.

26. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{17}$.

27. (a) (*Important*) Show that for $x \neq 1$

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

(b) Determine whether the sequence

$$a_n = \sum_{k=0}^{n} \frac{1}{3^k}$$

converges and, if it does, find the limit.

28. Express

$$\sum_{k=1}^{n} \frac{a_k}{10^k}$$

as a decimal fraction, given that each a_k is an integer from 0 to 9.

29. Let p be a positive integer. Show that, as $n \to \infty$,

$$a_n \to l \quad \text{iff} \quad a_{n-p} \to l.$$

30. Show that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}.$$

Verify by induction.

31. $\sum_{k=1}^{n} k = \frac{1}{2}(n)(n+1).$

32. $\sum_{k=1}^{n} (2k-1) = n^2.$

33. $\sum_{k=1}^{n} k^2 = \frac{1}{6}(n)(n+1)(2n+1).$

34. $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$

13.2 ····INFINITE SERIES

Introduction; Definitions

While it's possible to add two numbers, three numbers, a hundred numbers, or even a million numbers, it's impossible to add an infinite number of numbers. The theory of *infinite series* arose from attempts to circumvent this impossibility.

To form an infinite series we begin with an infinite sequence of real numbers: a_0, a_1, a_2, ... We can't form the sum of all the a_k (there is an infinite number of them), but we can form the *partial sums*

$$s_0 = a_0 = \sum_{k=0}^{0} a_k,$$

$$s_1 = a_0 + a_1 = \sum_{k=0}^{1} a_k,$$

$$s_2 = a_0 + a_1 + a_2 = \sum_{k=0}^{2} a_k,$$

$$s_3 = a_0 + a_1 + a_2 + a_3 = \sum_{k=0}^{3} a_k,$$

.

.

$$s_n = a_0 + a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

If the sequence $\{s_n\}$ of partial sums converges to a finite limit l, we write

$$\sum_{k=0}^{\infty} a_k = l$$

and say that

$$\text{the } \textit{series} \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges} \text{ to } l.$$

We call l the *sum* of the series. If the sequence of partial sums diverges, we say that

$$\text{the } \textit{series} \quad \sum_{k=0}^{\infty} a_k \quad \textit{diverges.}$$

REMARK. It is important to note that the sum of a series is not a sum in the ordinary sense. It is a limit. □

Here are some examples.

EXAMPLE. We begin with the series

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}.$$

To determine whether or not this series converges we must examine the partial sums.

Since

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2},$$

you can see that

$$\begin{aligned} s_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2}. \end{aligned}$$

Since all but the first and last terms occur in pairs with opposite signs, the sum collapses to give

$$s_n = 1 - \frac{1}{n+2}.$$

Obviously, as $n \to \infty$, $s_n \to 1$. This means that the series converges to 1:

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1. \quad \square$$

EXAMPLE. Here we examine two divergent series

$$\sum_{k=0}^{\infty} 2^k \qquad \text{and} \qquad \sum_{k=1}^{\infty} (-1)^k.$$

The partial sums of the first series take the form

$$s_n = \sum_{k=0}^{n} 2^k = 1 + 2 + \cdots + 2^n.$$

The sequence $\{s_n\}$ is unbounded and therefore divergent (12.2.5). This means that the series diverges.

For the second series we have

$$s_n = -1 \text{ if } n \text{ is odd} \qquad \text{and} \qquad s_n = 0 \text{ if } n \text{ is even.}$$

The sequence of partial sums looks like this:

$$-1, \quad 0, \quad -1, \quad 0, \quad -1, \quad 0, \quad \cdots .$$

The series diverges since the sequence of partial sums diverges. □

The Geometric Series

In school you probably studied geometric progression:

$$1, \quad x, \quad x^2, \quad x^3, \quad \cdots .$$

The sums

$$1, \quad 1+x, \quad 1+x+x^2, \quad 1+x+x^2+x^3, \quad \cdots$$

generated by numbers in geometric progression are the partial sums of what is known as the *geometric series*:

$$\sum_{k=0}^{\infty} x^k.$$

The geometric series arises in so many different contexts that it merits special attention.

The following result is fundamental:

(13.2.1)

(i) if $|x| < 1$, then $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$;

(ii) if $|x| \geq 1$, then $\displaystyle\sum_{k=0}^{\infty} x^k$ diverges.

Proof. The nth partial sum of the geometric series

$$\sum_{k=0}^{\infty} x^k$$

takes the form

(1) $$s_n = 1 + x + \cdots + x^n.$$

Multiplication by x gives

$$xs_n = x + x^2 + \cdots + x^{n+1}.$$

Subtracting the second equation from the first, we find that

$$(1 - x)s_n = 1 - x^{n+1}.$$

For $x \neq 1$, this gives

(2) $$s_n = \frac{1 - x^{n+1}}{1 - x}.$$

If $|x| < 1$, then $x^{n+1} \to 0$ and thus, by equation (2),

$$s_n \to \frac{1}{1 - x}.$$

This proves (i).

Now let's prove (ii). For $x = 1$, we use equation (1) and deduce that $s_n = n + 1$. Obviously, $\{s_n\}$ diverges. For $x \neq 1$ with $|x| \geq 1$, we use equation (2). Since in this instance $\{x^{n+1}\}$ diverges, $\{s_n\}$ diverges. □

You may have seen (13.2.1) before written as

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \begin{cases} \dfrac{a}{1-r}, & |r| < 1 \\ \text{diverges}, & |r| \geq 1 \end{cases}.$$

Taking $a = 1$ and $r = \frac{1}{2}$, we have

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1 - \frac{1}{2}} = 2.$$

Begin the summation at $k = 1$ instead of at $k = 0$, and you see that

(13.2.2)

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

The partial sums of this series

$$s_1 = \tfrac{1}{2},$$
$$s_2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4},$$
$$s_3 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8},$$
$$s_4 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{15}{16},$$
$$s_5 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} = \tfrac{31}{32},$$
etc.

are illustrated in Figure 13.2.1. Each new partial sum lies halfway between the previous partial sum and the number 1.

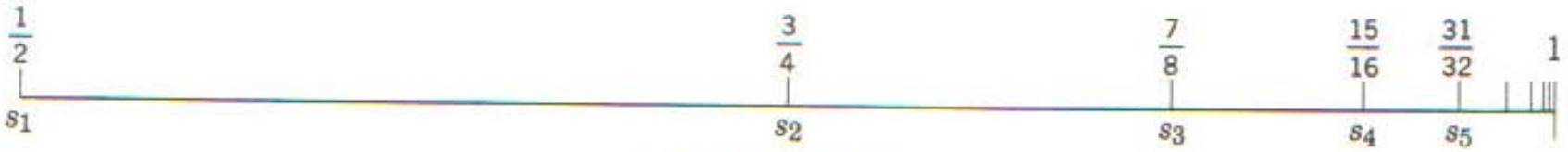

FIGURE 13.2.1

The convergence of the geometric series at $x = \frac{1}{10}$ enables us to assign a precise meaning to infinite decimals. Begin with the fact that

$$\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k = \sum_{k=0}^{\infty}\frac{1}{10^k} = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}.$$

This gives

$$\sum_{k=1}^{\infty}\frac{1}{10^k} = \frac{1}{9}$$

and shows that the partial sums

$$s_n = \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n}$$

are all less than $\frac{1}{9}$. Now take a series of the form

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k} \qquad \text{with} \quad a_k = 0, 1, \ldots, \text{or } 9.$$

Its partial sums

$$t_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$$

are all less than 1:

$$t_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} \le 9\left(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n}\right) = 9s_n < 9\left(\frac{1}{9}\right) = 1.$$

Since $\{t_n\}$ is nondecreasing, as well as bounded above, $\{t_n\}$ is convergent; this means that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k}$$

is convergent. The sum of this series is what we mean by the infinite decimal

$$0.a_1a_2a_3 \cdots a_n \cdots .$$

Below are two simple problems that lead naturally to the geometric series. You'll find more in the exercises.

PROBLEM. An electric fan is turned off, and the blades begin to lose speed. Given that the blades turn N times during the first second of no power and lose at least $\sigma\%$ of their speed with the passing of each ensuing second, show that the blades cannot turn more than $100N\sigma^{-1}$ times after power shutdown.

Solution. The number of turns during the first second of no power is

$$N.$$

The number of turns during the first 2 seconds is at most

$$N + \left(1 - \frac{1}{100}\sigma\right)N;$$

during the first 3 seconds, at most

$$N + \left(1 - \frac{1}{100}\sigma\right)N + \left(1 - \frac{1}{100}\sigma\right)^2 N;$$

and, during the first $n + 1$ seconds, at most

$$N \sum_{k=0}^{n} \left(1 - \frac{1}{100}\sigma\right)^k.$$

The total number of turns after power shutdown cannot exceed the limiting value

$$N \sum_{k=0}^{\infty} \left(1 - \frac{1}{100}\sigma\right)^k = N\left[\frac{1}{1 - (1 - \frac{1}{100}\sigma)}\right] = \frac{100N}{\sigma} = 100N\sigma^{-1}. \quad \square$$

PROBLEM. According to Figure 13.2.2, it is 2 o'clock. At what time between 2 and 3 o'clock will the two hands coincide?

Solution. We will solve the problem by setting up a geometric series. We will then confirm our answer by approaching the problem from a different perspective.

The hour hand travels one-twelfth as fast as the minute hand. At 2 o'clock the minute hand points to 12 and the hour hand points to 2. By the time the minute hand reaches 2, the hour hand points to $2 + \frac{1}{6}$. By the time the minute hand reaches

FIGURE 13.2.2

$2 + \frac{1}{6}$, the hour hand points to

$$2 + \frac{1}{6} + \frac{1}{6 \cdot 12}.$$

By the time the minute hand reaches

$$2 + \frac{1}{6} + \frac{1}{6 \cdot 12},$$

the hour hand points to

$$2 + \frac{1}{6} + \frac{1}{6 \cdot 12} + \frac{1}{6 \cdot 12^2}$$

and so on. In general, by the time the minute hand reaches

$$2+\frac{1}{6}+\frac{1}{6\cdot 12}+\cdots+\frac{1}{6\cdot 12^{n-1}}=2+\frac{1}{6}\sum_{k=0}^{n-1}\frac{1}{12^k},$$

the hour hand points to

$$2+\frac{1}{6}+\frac{1}{6\cdot 12}+\cdots+\frac{1}{6\cdot 12^{n-1}}+\frac{1}{6\cdot 12^{n}}=2+\frac{1}{6}\sum_{k=0}^{n}\frac{1}{12^k}.$$

The two hands coincide when they both point to the limiting value

$$2+\frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{12^k}=2+\frac{1}{6}\left(\frac{1}{1-\frac{1}{12}}\right)=2+\frac{2}{11}.$$

This happens at $2+\frac{2}{11}$ o'clock, approximately 10 minutes and 55 seconds after 2.

We can confirm this as follows. Suppose that the two hands meet at hour $2+x$. The hour hand will have moved a distance x and the minute hand a distance $2+x$. Since the minute hand moves 12 times as fast as the hour hand, we have

$$12x=2+x, \qquad 11x=2, \qquad \text{and thus} \qquad x=\frac{2}{11}. \quad \square$$

We'll return to the geometric series later. Right now we turn our attention to series in general.

Some Basic Results

(13.2.3)

1. If

$$\sum_{k=0}^{\infty} a_k \quad \text{converges} \qquad \text{and} \qquad \sum_{k=0}^{\infty} b_k \text{ converges,}$$

then

$$\sum_{k=0}^{\infty} (a_k + b_k) \quad \text{converges.}$$

Moreover, if

$$\sum_{k=0}^{\infty} a_k = l \qquad \text{and} \qquad \sum_{k=0}^{\infty} b_k = m,$$

then

$$\sum_{k=0}^{\infty} (a_k + b_k) = l + m.$$

2. If

$$\sum_{k=0}^{\infty} a_k \quad \text{converges,}$$

then

$$\sum_{k=0}^{\infty} \alpha a_k \quad \text{converges for each real number } \alpha.$$

Moreover,

$$\text{if} \quad \sum_{k=0}^{\infty} a_k = l, \qquad \text{then} \quad \sum_{k=0}^{\infty} \alpha a_k = \alpha l.$$

Proof. Let

$$s_n = \sum_{k=0}^{n} a_k, \qquad t_n = \sum_{k=0}^{n} b_k,$$

$$u_n = \sum_{k=0}^{n} (a_k + b_k), \qquad v_n = \sum_{k=0}^{n} \alpha a_k.$$

Note that

$$u_n = s_n + t_n \quad \text{and} \quad v_n = \alpha s_n.$$

If $s_n \to l$ and $t_n \to m$, we have

$$u_n \to l + m \quad \text{and} \quad v_n \to \alpha l. \quad \square$$

THEOREM 13.2.4

The *kth term* of a convergent series tends to 0; namely,

$$\text{if} \quad \sum_{k=0}^{\infty} a_k \quad \text{converges,} \qquad \text{then} \quad a_k \to 0.$$

Proof. To say that the series converges is to say that the sequence of partial sums converges to some number l:

$$s_n = \sum_{k=0}^{n} a_k \to l.$$

Obviously, then, $s_{n-1} \to l$. Since $a_n = s_n - s_{n-1}$, we have $a_n \to l - l = 0$. A change in notation gives $a_k \to 0$. □

The next result is an obvious, but important, consequence of Theorem 13.2.4.

THEOREM 13.2.5 A DIVERGENCE TEST

$$\text{If} \quad a_k \not\to 0, \qquad \text{then} \quad \sum_{k=0}^{\infty} a_k \quad \text{diverges.}$$

EXAMPLES. Since $\dfrac{k}{k+1} \not\to 0$, the series

$$\sum_{k=0}^{\infty} \frac{k}{k+1} = 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

diverges. Since $\sin k \not\to 0$, the series

$$\sum_{k=0}^{\infty} \sin k = \sin 0 + \sin 1 + \sin 2 + \sin 3 + \cdots$$

also diverges. □

CAUTION. Theorem 13.2.4 does *not* say that, if $a_k \to 0$, then $\sum_{k=0}^{\infty} a_k$ converges.

There are divergent series for which $a_k \to 0$.

EXAMPLE. In the case of

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

we have

$$a_k = \frac{1}{\sqrt{k}} \to 0,$$

but, since

$$s_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \frac{n}{\sqrt{n}} = \sqrt{n},$$

the sequence of partial sums is unbounded, and therefore the series diverges. □

EXERCISES

Find the sum of the series.

1. $\sum_{k=3}^{\infty} \frac{1}{(k+1)(k+2)}$.
2. $\sum_{k=0}^{\infty} \frac{1}{(k+3)(k+4)}$.
3. $\sum_{k=1}^{\infty} \frac{1}{2k(k+1)}$.
4. $\sum_{k=3}^{\infty} \frac{1}{k^2-k}$.
5. $\sum_{k=1}^{\infty} \frac{1}{k(k+3)}$.
6. $\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+3)}$.
7. $\sum_{k=0}^{\infty} \frac{3}{10^k}$.
8. $\sum_{k=0}^{\infty} \frac{12}{100^k}$.
9. $\sum_{k=0}^{\infty} \frac{67}{1000^k}$.
10. $\sum_{k=0}^{\infty} \frac{(-1)^k}{5^k}$.
11. $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$.
12. $\sum_{k=0}^{\infty} \frac{3^k + 4^k}{5^k}$.
13. $\sum_{k=0}^{\infty} \frac{1-2^k}{3^k}$.
14. $\sum_{k=0}^{\infty} \left(\frac{25}{10^k} - \frac{6}{100^k}\right)$.
15. $\sum_{k=3}^{\infty} \frac{1}{2^{k-1}}$.
16. $\sum_{k=0}^{\infty} \frac{1}{2^{k+3}}$.
17. $\sum_{k=0}^{\infty} \frac{2^{k+3}}{3^k}$.
18. $\sum_{k=2}^{\infty} \frac{3^{k-1}}{4^{3k+1}}$.

Write the decimal fraction as an infinite series and express the sum as the quotient of two integers.

19. $0.777\cdots$.

20. $0.999\cdots$.

21. $0.\overbrace{24}\overbrace{24}\cdots$.

22. $0.\overbrace{89}\overbrace{89}\cdots$.

23. $0.\overbrace{112}\overbrace{112}\overbrace{112}\cdots$.

24. $0.\overbrace{315}\overbrace{315}\overbrace{315}\cdots$.

25. $0.62\overbrace{45}\overbrace{45}\cdots$.

26. $0.112\overbrace{019}\overbrace{019}\cdots$.

27. Using series, show that every repeating decimal represents a rational number (the quotient of two integers).

28. Show that

$$\sum_{k=0}^{\infty} a_k = l \quad \text{iff} \quad \sum_{k=j+1}^{\infty} a_k = l - (a_0 + \cdots + a_j).$$

Derive these results from the geometric series.

29. $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}, \quad |x| < 1.$

30. $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}, \quad |x| < 1.$

Find a series expansion for the given expression.

31. $\dfrac{x}{1-x}$ valid for $|x| < 1$.

32. $\dfrac{x}{1+x}$ valid for $|x| < 1$.

33. $\dfrac{x}{1+x^2}$ valid for $|x| < 1$.

34. $\dfrac{1}{4-x^2}$ valid for $|x| < 2$.

35. $\dfrac{1}{1+4x^2}$ valid for $|x| < \dfrac{1}{2}$.

36. $\dfrac{x^2}{1-x}$ valid for $|x| < 1$.

37. At some time between 4 and 5 o'clock the minute hand is directly above the hour hand. Express this time as a geometric series. What is the sum of this series?

38. Given that a ball dropped to the floor rebounds to a height proportional to the height from which it is dropped, find the total distance traveled by a ball dropped from a height of 6 feet if it rebounds initially to a height of 3 feet.

39. Exercise 38 under the supposition that the ball rebounds initially to a height of 2 feet.

40. In the setting of Exercise 38, to what height does the ball rebound initially if the total distance traveled by the ball is 21 feet?

41. How much money must you deposit at r% interest compounded annually to enable your descendants to withdraw n_1 dollars at the end of the first year, n_2 dollars at the end of the second year, n_3 dollars at the end of the third year, and so on in perpetuity? Assume that the sequence $\{n_k\}$ is bounded, $n_k \leq N$ for all k, and express your answer as an infinite series.

42. Sum the series you obtained in Exercise 41 setting

(a) $r = 5$, $n_k = 5000(\frac{1}{2})^{k-1}$. (b) $r = 6$, $n_k = 1000(0.8)^{k-1}$. (c) $r = 5$, $n_k = N$.

43. Show that the series

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$$

diverges even though $\lim_{k\to\infty} \ln\left(\frac{k+1}{k}\right) = 0$.

44. Show that the series

$$\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^k$$

diverges.

45. Let $\{d_k\}$ be a sequence of real numbers that converges to 0.

(a) Show that

$$\sum_{k=1}^{\infty} (d_k - d_{k+1}) = d_1.$$

(b) Sum the following series:

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k(k+1)}}.$ (ii) $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{2k^2(k+1)^2}.$

46. Show that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \quad \text{for } |x| < 1.$$

HINT: Verify that s_n, the nth partial sum, satisfies the identity

$$(1-x)^2 s_n = 1 - (n+1)x^n + nx^{n+1}.$$

13.3 ···· THE INTEGRAL TEST; COMPARISON THEOREMS

Here and in the next section we direct our attention to *series with nonnegative terms*: $a_k \geq 0$ for all k. For such series the following theorem is fundamental.