Calculus Exploration 2: Algebraic Limit Rules

Constant Multiple Rule (CMR)

$$\lim_{x\to a} c\,\mathrm{f}(x) = c\,(\lim_{x\to a} \mathrm{f}(x))$$

Sum of Functions Rule (SR)

$$\lim_{x\to a} \;(\mathrm{f}(x) + \mathrm{g}(x)) = (\lim_{x\to a} \mathrm{f}(x)) + (\lim_{x\to a} \mathrm{g}(x))$$

Difference of Functions Rule (DR)

$$\lim_{x\to a} \;(\mathrm{f}(x) - \mathrm{g}(x)) = (\lim_{x\to a} \mathrm{f}(x)) - (\lim_{x\to a} \mathrm{g}(x))$$

Product of Functions Rule (PR)

$$\lim_{x\to a} \mathrm{f}(x)\,\mathrm{g}(x) = (\lim_{x\to a} \mathrm{f}(x))\,(\lim_{x\to a} \mathrm{g}(x))$$

Quotient of Functions Rule (QR)

$$\lim_{x\to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \frac{\lim_{x\to a} \mathrm{f}(x)}{\lim_{x\to a} \mathrm{g}(x)}$$

Constant Function Rule (CFR)

$$\textit{If for all } x,\;\; \mathrm{f}(x) = c,\;\; \textit{then }\; (\lim_{x\to a} \mathrm{f}(x)) = (\lim_{x\to a} c) \;=\; c$$

Identity Function Rule (IFR)

$$\textit{If for all } x,\;\; \mathrm{f}(x) = x,\;\; \textit{then }\; (\lim_{x\to a} \mathrm{f}(x)) = (\lim_{x\to a} x) \;=\; a$$

Power Function Rule 1 (PFR 1)

$$\textit{If for all } x,\;\; \mathrm{f}(x) = x^n,\;\; \textit{then }\; (\lim_{x\to a} \mathrm{f}(x)) \;=\; (\lim_{x\to a} x^n) = (\lim_{x\to a} x)^n$$

Power Function Rule 2 (PFR 2)

$$\textit{If for all } x,\;\; \mathrm{g}(x) = \mathrm{f}(x)^n,\;\; \textit{then }\; (\lim_{x\to a} \mathrm{g}(x)) \;=\; (\lim_{x\to a} \mathrm{f}(x)^n) = (\lim_{x\to a} \mathrm{f}(x))^n$$

Power Function Rule 3 (PFR 3)

$$\textit{If for all } x,\;\; \mathrm{h}(x) = \mathrm{f}(x)^{\mathrm{g}(x)},\;\; \textit{then }\; (\lim_{x\to a} \mathrm{h}(x)) \;=\; (\lim_{x\to a} \mathrm{f}(x)^{\mathrm{g}(x)}) = (\lim_{x\to a} \mathrm{f}(x))^{(\lim_{x\to a} \mathrm{g}(x))}$$

Algebraic Limit Rules: Example 1

Next, we will carefully evaluate the following limit, using the limit rules listed on the previous page, and justifying each step by labeling each one with the appropriate rule(s) that is (are) used.

$$\lim_{x\to 1}\ (x^2+3\,x+4) = (\lim_{x\to 1} x^2) + (\lim_{x\to 1} 3\,x) + (\lim_{x\to 1} 4),\ \ \textit{By SR}$$
$$= (\lim_{x\to 1} x)^2 + (\lim_{x\to 1} 3\,x) + (\lim_{x\to 1} 4),\ \ \textit{By PFR 1}$$
$$= 1 + (\lim_{x\to 1} 3\,x) + (\lim_{x\to 1} 4),\ \ \textit{By IFR}$$
$$= 5 + (\lim_{x\to 1} 3\,x),\ \ \textit{By CFR}$$
$$= 5 + 3\,(\lim_{x\to 1} x),\ \ \textit{By CMR}$$
$$= 8,\ \ \textit{By IFR}$$
$$\textit{So}\ ,\ \lim_{x\to 1}\ (x^2+3\,x+4) = 8$$

Note that this is an example of the (by now) familiar fact that

if f *is a continuous function at* $x = a$, *then* $(\lim_{x\to a} \mathrm{f}(x)) = \mathrm{f}(a)$

Algebraic Limit Rules: Example 2

Again, we will carefully evaluate the following limit, using the limit rules listed on the previous page, and justifying each step by labeling each one with the appropriate rule(s) that is (are) used.

$$\lim_{x\to 2} \frac{x^2-1}{x-1} = \frac{\lim_{x\to 2} x^2-1}{\lim_{x\to 2} x-1},\ \ \textit{By QR}$$
$$= \frac{(\lim_{x\to 2} x^2) + (\lim_{x\to 2}\ -1)}{(\lim_{x\to 2} x) + (\lim_{x\to 2}\ -1)},\ \ \textit{By SR}$$
$$= \frac{3}{(\lim_{x\to 2} x) + (\lim_{x\to 2}\ -1)},\ \ \textit{By PFR 1},\ \textit{IFR},\ \textit{CFR}$$
$$= 3,\ \ \textit{By IFR},\ \textit{CFR}$$
$$\textit{So}\ ,\ \lim_{x\to 2} \frac{x^2-1}{x-1} = 3$$

Note that this is another example of the (by now) familiar fact that

if f *is a continuous function at* $x = a$, *then* $(\lim_{x\to a} \mathrm{f}(x)) = \mathrm{f}(a)$

Algebraic Limit Rules: Example 3

In this example, we will use the same function as in Example 2, but take the limit as x approaches 1, instead. Notice that if we try to proceed as in Example 2 we have:

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \frac{\lim\limits_{x\to 1} x^2-1}{\lim\limits_{x\to 1} x-1}, \quad \textit{By QR ??}$$

If we try to continue as in Example 2, we will end up with 0 in the denominator. This illustrates that we can only apply the algebraic limit rules when all of the component limits exist, and when we don't introduce division by zero. So, we can't apply the Quotient Rule in this example. This means we need a different approach to evaluate this limit (if it exists). The key is to try a little algebra:

$$(\lim_{x\to 1}\frac{x^2-1}{x-1}) = (\lim_{x\to 1}\frac{(x-1)\,(x+1)}{x-1}) = (\lim_{x\to 1} x+1) = \textit{2}, \quad \textit{By SR, IFR, CFR}$$

Notice that this is an example of a function for which the limit exists as x approaches 1, but 1 isn't in the domain of the function. What does the graph of this function look like?

Algebraic Limit Rules: Example 4

In this example, we will use the reciprocal of the function in Example 3, and take the limit as x approaches -1. In addition, we try to use the same algebraic simplification, however in this case the limit does not exist. What does the graph of this function look like?

$$(\lim_{x\to(-1)}\frac{x-1}{x^2-1}) = (\lim_{x\to(-1)}\frac{x-1}{(\,x-1)\,(\,x+1)}) = (\lim_{x\to(-1)}\frac{1}{x+1}) \quad \textit{which doesn't exist.}$$

Notice that this function diverges in two different ways as x approaches -1 from the left or from the right.

$$(\lim_{x\to(-1)-}\frac{x-1}{x^2-1}) = (\lim_{x\to(-1)-}\frac{x-1}{(\,x-1)\,(\,x+1)}) = (\lim_{x\to(-1)-}\frac{1}{x+1}) = -\infty$$

$$(\lim_{x\to(-1)+}\frac{x-1}{x^2-1}) = (\lim_{x\to(-1)+}\frac{x-1}{(\,x-1)\,(\,x+1)}) = (\lim_{x\to(-1)+}\frac{1}{x+1}) = \infty$$

Algebraic Limit Rules: Exercises

In each exercise below, if possible, find the limit and list all the limit rules used. Otherwise, explain why the limit does not exist.

1., $\lim_{x \to 4} 2 =$

2., $\lim_{x \to 3} (5 - 4\,x)^2 =$

3., $\lim_{x \to (-4)} x^2 + 3\,x - 7 =$

4., $\lim_{x \to 2} \frac{3\,x}{x + 4} =$

5., $\lim_{x \to (-1)} \frac{x^2 + 1}{3\,x^5 + 4} =$

6., $\lim_{x \to 0} \frac{x^2}{x^2 + 1} =$

7., $\lim_{x \to 0} \frac{x^2 + 1}{x^2} =$

8., $\lim_{x \to 2} \frac{x}{x^2 - 4} =$

9., $\lim_{h \to 0} h\,(1 - \frac{1}{h}) =$

10., $\lim_{h \to 0} \frac{1 - \frac{1}{h}}{1 - \frac{2}{h}} =$

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