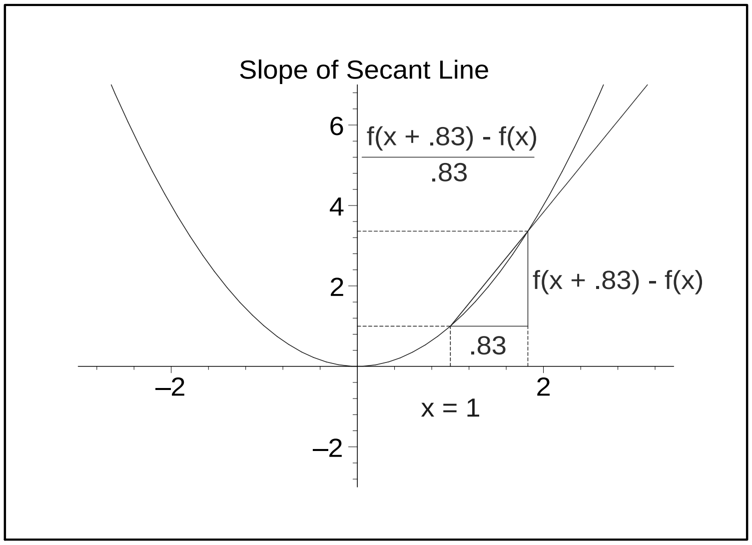
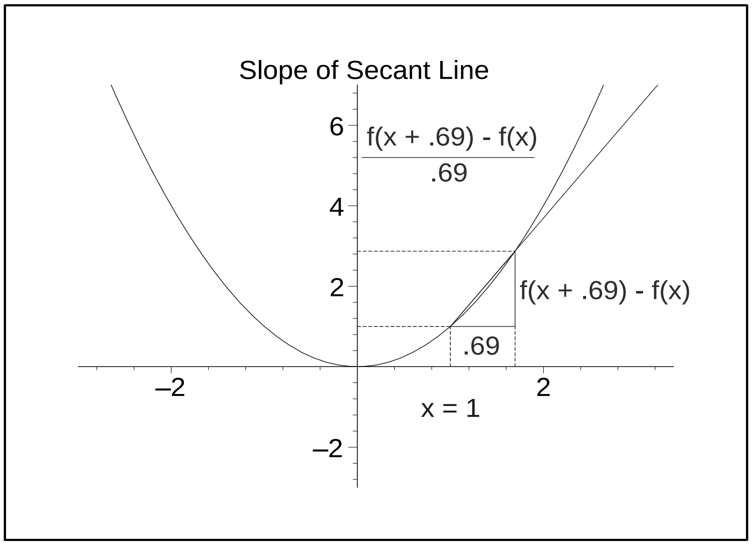
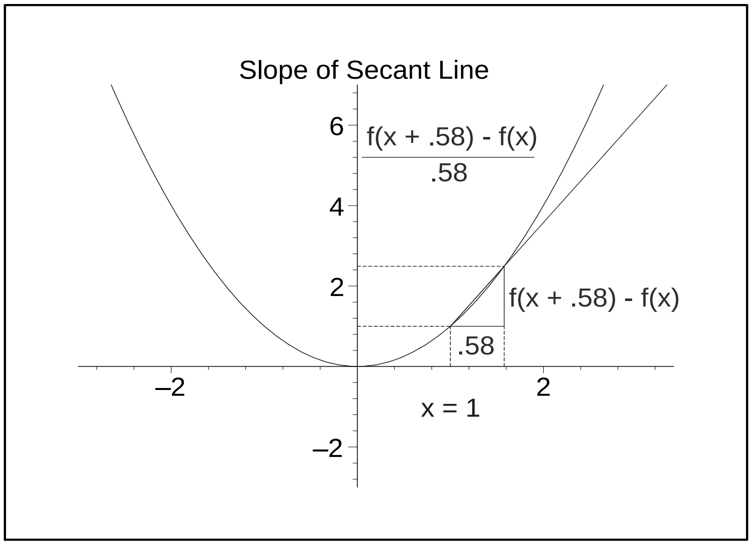
Calculus Exploration 3:

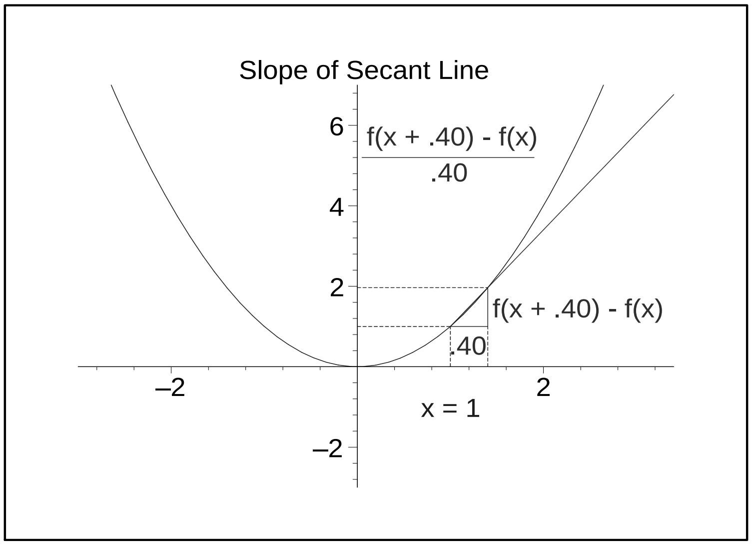
Tangents as Limits of Secant Lines

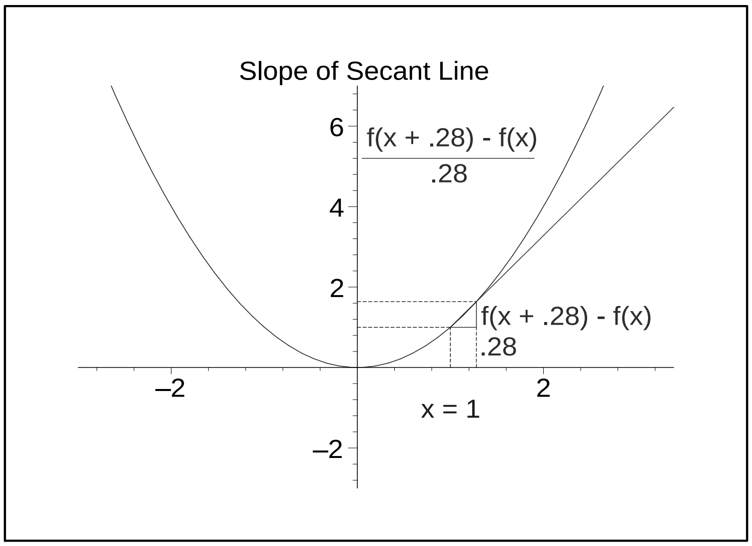
In Example 1, we approximate the slope of the tangent line to $\mathrm{f}(x) = x^2$, *at* $x = 1$

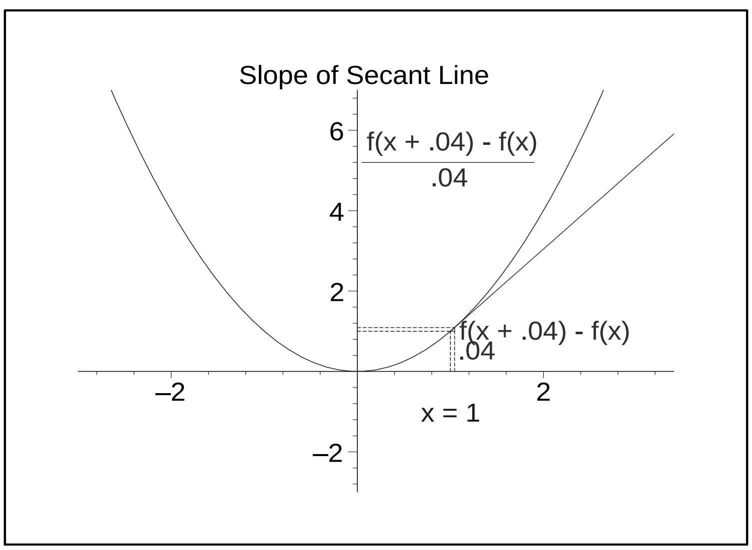












For the function $\mathrm{f}(x) = x^2$, $x = 1$, *the six graphs above provide us with the following sequence of approximations to the slope of the tangent line :*

$$\frac{\mathrm{f}(x+0.83)-\mathrm{f}(x)}{.83} = 2.831,\ \frac{\mathrm{f}(x+0.69)-\mathrm{f}(x)}{.69} = 2.689,\ \frac{\mathrm{f}(x+0.58)-\mathrm{f}(x)}{.58} = 2.579$$

$$\frac{f(x+0.40)-f(x)}{.40}=2.400,\ \frac{f(x+0.28)-f(x)}{.28}=2.278,\ \frac{f(x+0.04)-f(x)}{.04}=2.05$$

$$\textit{Apparently, if } f(x)=x^2,\ x=1,\ \textit{then, } \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=2$$

$$\textit{Let's see if we can prove this, but first a little algebra:}$$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{(x+h)^2-x^2}{h}\\&=\frac{x^2+2\,x\,h+h^2-\ x^2}{h}\\&=\frac{2\,x\,h+h^2}{h}\\&=2\,x+h\end{aligned}$$

$$\textit{So, we have shown that}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}2\,x+h$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=2\,x$$

$$\textit{Thus, 2 is the slope of the tangent line to } f(x)=x^2,\ \textit{at } x=1$$

$$\textit{Notice that we have found a general formula for the slope}$$

$$\textit{of the tangent line to } f(x)=x^2,\ \textit{that works for any } x\ !$$

$$\textit{Definition of Tangent Slope Function (Derivative):}$$

$$tsf(x)\ =\ f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Tangents as Limits of Secant Lines: Exercises

In each exercise 1-3 below, find the tangent slope function (derivative) for each function by taking the limit. Hint: Use the Binomial Theorem or Pascal's triangle. Also, after you complete these exercises, describe any pattern you see by trying to capture it in one clear sentence that you write down.

1. $f(x)=x^3,\ f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

So, to get started $f'(x)=\lim_{h\to 0}\dfrac{(x+h)^3-x^3}{h}$

2. $\mathrm{f}(x) = x^3 - x$, $\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$

So, to get started $\mathrm{f}'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x - h - \ x^3 + \ x}{h}$

3. $\mathrm{f}(x) = -x^3 + 3\,x + 2$, $\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$

So, to get started $\mathrm{f}'(x) = \lim_{h \to 0} \frac{-(x+h)^3 + 3\,x + 3\,h + \ x^3 - 3\ \ x}{h}$

4. The plot below contains the graph of the function in Exercise 3 above. Use the tangent slope function found there to find the equation of the tangent line to the curve below at each of the following values: x = -2, -1.6, -1, 0, 1, 1.6, 2. Also, accurately sketch each tangent line on the plot below.