Calculus Exploration 4: Derivative Rules (D-Rules)

Constant Multiple Rule (CMR)

$$\mathrm{D}\,[c\,\mathrm{f}(x)] = c\,\mathrm{D}\,[\mathrm{f}(x)]$$

Sum of Functions Rule (SR)

$$\mathrm{D}\,[\mathrm{f}(x) + \mathrm{g}(x)] = \mathrm{D}\,[\mathrm{f}(x)] + \mathrm{D}\,[\mathrm{g}(x)]$$

Constant Function Rule (CFR)

$$\textit{If for all } x,\;\; \mathrm{f}(x) = c,\;\; \textit{where } c \textit{ is a constant number},\;\; \textit{then}\;\; \mathrm{D}\,[\mathrm{f}(x)] = \mathrm{D}\,[c] \;=\; 0$$

Identity Function Rule (IFR)

$$\textit{If for all } x,\;\; \mathrm{f}(x) = x,\;\; \textit{then}\;\; \mathrm{D}\,[\mathrm{f}(x)] = \mathrm{D}\,[x] \;=\; 1$$

Power Function Rule (PFR)

$$\textit{If for all } x,\;\; \mathrm{f}(x) = x^n,\;\; \textit{then}\;\; \mathrm{D}\,[\mathrm{f}(x)] = \mathrm{D}\,[x^n] \;=\; n\,x^{(n-1)}$$

Product of Functions Rule (PR)

$$\mathrm{D}\,[\mathrm{f}(x)\,\mathrm{g}(x)] = \mathrm{D}\,[\mathrm{f}(x)]\,\mathrm{g}(x) + \mathrm{D}\,[\mathrm{g}(x)]\,\mathrm{f}(x)$$

Quotient of Functions Rule (QR)

$$\mathrm{D}\,[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}] = \frac{\mathrm{D}\,[\mathrm{f}(x)]\,\mathrm{g}(x) - \mathrm{D}\,[\mathrm{g}(x)]\,\mathrm{f}(x)}{\mathrm{g}(x)^2}$$

Power Chain Rule (PowCR)

$$\mathrm{D}\,[\mathrm{f}(x)^n] = n\,\mathrm{f}(x)^{(n-1)}\,\mathrm{D}\,[\mathrm{f}(x)]$$
$$\textit{Note the special case}:\; \mathrm{D}\,[x^n] = n\,x^{(n-1)}\,\mathrm{D}\,[x]$$

Exponential Rule (ExpR) & Exponential Chain Rule (ExpCR)

$$\mathrm{D}\,[e^x] = e^x \qquad (\; \textit{ExpR}\;)$$
$$\mathrm{D}\,[e^{\mathrm{f}(x)}] = e^{\mathrm{f}(x)}\,\mathrm{D}\,[\mathrm{f}(x)] \qquad (\; \textit{ExpCR}\;)$$

Natural Log Rule (LnR) & Natural Log Chain Rule (LnCR)

$$\mathrm{D}\,[\ln(x)] = \frac{1}{x}, \qquad (\; \textit{LnR}\;)$$
$$\mathrm{D}\,[\ln(\mathrm{f}(x))] = \frac{\mathrm{D}\,[\mathrm{f}(x)]}{\mathrm{f}(x)}, \qquad (\; \textit{LnCR}\;)$$

Sine Rule (SinR) & Sine Chain Rule (SinCR)

$$\mathrm{D}\left[\sin(x)\right] = \cos(x) \quad (\ \textit{SinR}\ )$$
$$\mathrm{D}\left[\sin(\mathrm{f}(x))\right] = \cos(\mathrm{f}(x))\,\mathrm{D}\left[\mathrm{f}(x)\right] \quad (\ \textit{SinCR}\ )$$

Cosine Rule (CosR) & Cosine Chain Rule (CosCR)

$$\mathrm{D}\left[\cos(x)\right] = -\sin(x) \quad (\ \textit{CosR}\ )$$
$$\mathrm{D}\left[\cos(\mathrm{f}(x))\right] = -\sin(\mathrm{f}(x))\,\mathrm{D}\left[\mathrm{f}(x)\right] \quad (\ \textit{CosCR}\ )$$

Tangent Rule (TanR) & Tangent Chain Rule (TanCR)

$$\mathrm{D}\left[\tan(x)\right] = \sec(x)^2 \quad (\ \textit{TanR}\ )$$
$$\mathrm{D}\left[\tan(\mathrm{f}(x))\right] = \sec(\mathrm{f}(x))^2\,\mathrm{D}\left[\mathrm{f}(x)\right] \quad (\ \textit{TanCR}\ )$$

Cotangent Rule (CotR) & Cotangent Chain Rule (CotCR)

$$\mathrm{D}\left[\cot(x)\right] = -\csc(x)^2 \quad (\ \textit{CotR}\ )$$
$$\mathrm{D}\left[\cot(\mathrm{f}(x))\right] = -\csc(\mathrm{f}(x))^2\,\mathrm{D}\left[\mathrm{f}(x)\right] \quad (\ \textit{CotCR}\ )$$

Secant Rule (SecR) & Secant Chain Rule (SecCR)

$$\mathrm{D}\left[\sec(x)\right] = \sec(x)\tan(x) \quad (\ \textit{SecR}\ )$$
$$\mathrm{D}\left[\sec(\mathrm{f}(x))\right] = \sec(\mathrm{f}(x))\tan(\mathrm{f}(x))\,\mathrm{D}\left[\mathrm{f}(x)\right] \quad (\ \textit{SecCR}\ )$$

Cosecant Rule (CscR) & Cosecant Chain Rule (CscCR)

$$\mathrm{D}\left[\csc(x)\right] = -\csc(x)\cot(x) \quad (\ \textit{CscR}\ )$$
$$\mathrm{D}\left[\csc(\mathrm{f}(x))\right] = -\csc(\mathrm{f}(x))\cot(\mathrm{f}(x))\,\mathrm{D}\left[\mathrm{f}(x)\right] \quad (\ \textit{CscCR}\ )$$

Examples Using the Product Rule (PR)

$$\mathrm{D}\,[\mathrm{f}(x)\,\mathrm{g}(x)] = \mathrm{D}\,[\mathrm{f}(x)]\,\mathrm{g}(x) + \mathrm{D}\,[\mathrm{g}(x)]\,\mathrm{f}(x)$$

Example 1.

$$\mathrm{D}\,[\mathrm{f}(x)\,\mathrm{g}(x)] = \mathrm{D}\,[\mathrm{f}(x)]\,\mathrm{g}(x) + \mathrm{D}\,[\mathrm{g}(x)]\,\mathrm{f}(x)$$
$$\mathrm{f}(x) = 9\,x^2 + 7\,x, \quad \mathrm{g}(x) = x^2 - 1$$

$$\begin{gathered}
\mathrm{D}\,[\mathrm{f}(x)\,\mathrm{g}(x)] = \\
\mathrm{D}\,[(9\,x^2 + 7\,x)\,(x^2 - 1)] = \\
\mathrm{D}(9\,x^2 + 7\,x)\,(x^2 - 1) + \mathrm{D}(x^2 - 1)\,(9\,x^2 + 7\,x) = \\
(18\,x + 7)\,(x^2 - 1) + 2\,x\,(9\,x^2 + 7\,x) = \\
18\,x^3 - 18\,x + 7\,x^2 - 7 + \\
18\,x^3 + 14\,x^2 = \\
36\,x^3 + 21\,x^2 - 18\,x - 7
\end{gathered}$$

Example 2.

$$\mathrm{D}\,[\mathrm{f}(x)\,\mathrm{g}(x)] = \mathrm{D}\,[\mathrm{f}(x)]\,\mathrm{g}(x) + \mathrm{D}\,[\mathrm{g}(x)]\,\mathrm{f}(x)$$
$$\mathrm{f}(x) = x^2 + 3\,x - 5, \quad \mathrm{g}(x) = x^3 + 6\,x^2 - 2\,x + 1$$

$$\begin{gathered}
\mathrm{D}\,[\mathrm{f}(x)\,\mathrm{g}(x)] = \\
\mathrm{D}\,[(x^2 + 3\,x - 5)\,(x^3 + 6\,x^2 - 2\,x + 1)] = \\
\mathrm{D}(x^2 + 3\,x - 5)\,(x^3 + 6\,x^2 - 2\,x + 1) + \mathrm{D}(x^3 + 6\,x^2 - 2\,x + 1)\,(x^2 + 3\,x - 5) = \\
(2\,x + 3)\,(x^3 + 6\,x^2 - 2\,x + 1) + (3\,x^2 + 12\,x - 2)\,(x^2 + 3\,x - 5) = \\
2\,x^4 + 15\,x^3 + 14\,x^2 - 4\,x + 3 + \\
3\,x^4 + 21\,x^3 + 19\,x^2 - 66\,x + 10 = \\
5\,x^4 + 36\,x^3 + 33\,x^2 - 70\,x + 13
\end{gathered}$$

Motivation: A Not Product Rule

We know from the Sum Rule that $\mathrm{D}(x + x^2) = \mathrm{D}(x) + \mathrm{D}(x^2)$

So, maybe products work like this too, possibly $\mathrm{D}(\;x\,x^2) = \mathrm{D}(x)\,\mathrm{D}(x^2)$??

But by the Power Rule, $3\,x^2 \;=\; \mathrm{D}(x^3) = \mathrm{D}(\;x\,x^2)$, *and* $\mathrm{D}(x)\,\mathrm{D}(x^2) = 2\,x\,.$

So, $\mathrm{D}(\;x\,x^2) \neq \mathrm{D}(x)\,\mathrm{D}(x^2)$, *and therefore the simple rule above can't be true* !

And we can see the more complicated Product Rule hiding in this example :

$$\begin{aligned}
\mathrm{D}(\;x\,x^2) = \;\; & \mathit{3}\,x^2 = \;\mathit{1}\,x^2 + 2\,x\;\;x \\
& = \mathrm{D}(x)\,x^2 + \mathrm{D}(x^2)\,x
\end{aligned}$$

Examples Using the Quotient Rule (QR)

$$\mathrm{D}\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right) = \frac{\mathrm{D}(\mathrm{f}(x))\,\mathrm{g}(x) - \mathrm{D}(\mathrm{g}(x))\,\mathrm{f}(x)}{\mathrm{g}(x)^2}$$

Example 1.

$$\mathrm{D}\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right) = \frac{\mathrm{D}(\mathrm{f}(x))\,\mathrm{g}(x) - \mathrm{D}(\mathrm{g}(x))\,\mathrm{f}(x)}{\mathrm{g}(x)^2}$$

$$\mathrm{f}(x) = 9\,x^2 + 7\,x, \quad \mathrm{g}(x) = x^2 - 1$$

$$\mathrm{D}\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right) = \mathrm{D}\left(\frac{9\,x^2 + 7\,x}{x^2 - 1}\right) =$$

$$\frac{\mathrm{D}(9\,x^2 + 7\,x)\,(x^2 - 1) - \mathrm{D}(x^2 - 1)\,(9\,x^2 + 7\,x)}{(x^2 - 1)^2} =$$

$$\frac{(18\,x + 7)\,(x^2 - 1) - 2\,x\,(9\,x^2 + 7\,x)}{(x^2 - 1)^2} =$$

$$\frac{(18\,x^3 - 18\,x + 7\,x^2 - 7) \,+\, (-18\,x^3 - 14\,x^2)}{(x^2 - 1)^2} =$$

$$\frac{-7\,x^2 - 18\,x - 7}{(x^2 - 1)^2}$$

Example 2.

$$\mathrm{D}\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right) = \frac{\mathrm{D}(\mathrm{f}(x))\,\mathrm{g}(x) - \mathrm{D}(\mathrm{g}(x))\,\mathrm{f}(x)}{\mathrm{g}(x)^2}$$

$$\mathrm{f}(x) = x^2 + 3\,x - 5, \quad \mathrm{g}(x) = x^3 + 6\,x^2 - 2\,x + 1$$

$$\mathrm{D}\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right) = \mathrm{D}\left(\frac{x^2 + 3\,x - 5}{x^3 + 6\,x^2 - 2\,x + 1}\right) =$$

$$\frac{\mathrm{D}(x^2 + 3\,x - 5)\,(x^3 + 6\,x^2 - 2\,x + 1) - \mathrm{D}(x^3 + 6\,x^2 - 2\,x + 1)\,(x^2 + 3\,x - 5)}{(x^3 + 6\,x^2 - 2\,x + 1)^2} =$$

$$\frac{(2\,x + 3)\,(x^3 + 6\,x^2 - 2\,x + 1) - (3\,x^2 + 12\,x - 2)\,(x^2 + 3\,x - 5)}{(x^3 + 6\,x^2 - 2\,x + 1)^2} =$$

$$\frac{(2\,x^4 + 15\,x^3 + 14\,x^2 - 4\,x + 3) \,+\, (-3\,x^4 - 21\,x^3 - 19\,x^2 + 66\,x - 10)}{(x^3 + 6\,x^2 - 2\,x + 1)^2} =$$

$$\frac{-x^4 - 6\,x^3 - 5\,x^2 + 62\,x - 7}{(x^3 + 6\,x^2 - 2\,x + 1)^2}$$

Motivation for QR: Try Some Algebra and Rules You Already Know

$$\textit{For } g(x) \textit{ nonzero, we know that, } \frac{\mathrm{g}(x)}{\mathrm{g}(x)} = 1$$

$$\text{So, the lefthand side is just } \mathrm{g}(x), \textit{ times }, \frac{1}{\mathrm{g}(x)}, \textit{ and}$$

$$\textit{by PR and CFR, whatever } \mathrm{D}\left(\frac{1}{\mathrm{g}(x)}\right) \textit{ is, it must satisfy}:$$

$$\mathrm{D}(\mathrm{g}(x))\ \frac{1}{\mathrm{g}(x)} + \mathrm{D}\left(\frac{1}{\mathrm{g}(x)}\right)\mathrm{g}(x) = \mathrm{D}(1) = 0$$

$$\mathrm{D}\left(\frac{1}{\mathrm{g}(x)}\right)\mathrm{g}(x) = -\frac{\mathrm{D}(\mathrm{g}(x))}{\mathrm{g}(x)}$$

$$\mathrm{D}\left(\frac{1}{\mathrm{g}(x)}\right) = -\frac{\mathrm{D}(\mathrm{g}(x))}{\mathrm{g}(x)^2}$$

$$\textit{Now, use this and PR again to get } \mathrm{D}\left(\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right) \textit{ in a similar way}:$$

$$\mathrm{D}\left(\mathrm{f}(x)\ \frac{1}{\mathrm{g}(x)}\right) = \mathrm{D}(\mathrm{f}(x))\ \frac{1}{\mathrm{g}(x)} + \mathrm{D}\left(\frac{1}{\mathrm{g}(x)}\right)\mathrm{f}(x)$$

$$= \frac{\mathrm{D}(\mathrm{f}(x))}{\mathrm{g}(x)} - \frac{\mathrm{D}(\mathrm{g}(x))\,\mathrm{f}(x)}{\mathrm{g}(x)^2}$$

$$= \frac{\mathrm{D}(\mathrm{f}(x))\,\mathrm{g}(x) - \mathrm{D}(\mathrm{g}(x))\,\mathrm{f}(x)}{\mathrm{g}(x)^2}$$

Examples Using the Power Chain Rule (PowCR)

$$\mathrm{D}(\mathrm{f}(x)^n) = n\,\mathrm{f}(x)^{(n-1)}\,\mathrm{D}(\mathrm{f}(x))$$

Example 1.

$$\mathrm{D}(\mathrm{f}(x)^n) = n\,\mathrm{f}(x)^{(n-1)}\,\mathrm{D}(\mathrm{f}(x))$$
$$\mathrm{f}(x) = 2\,x^3 - 5\,x + 1, \quad n = 10$$

$$\mathrm{D}(\mathrm{f}(x)^n) = \mathrm{D}((2\,x^3 - 5\,x + 1)^{10}) \;=$$
$$10\,(2\,x^3 - 5\,x + 1)^9\,\mathrm{D}(2\,x^3 - 5\,x + 1) =$$
$$10\,(2\,x^3 - 5\,x + 1)^9\,(6\,x^2 - 5)$$

Example 2.

$$\mathrm{D}(\mathrm{f}(x)^n) = n\,\mathrm{f}(x)^{(n-1)}\,\mathrm{D}(\mathrm{f}(x))$$
$$\mathrm{f}(x) = x^2 + 3\,x - 5, \quad n = \frac{4}{3}$$

$$\mathrm{D}(\mathrm{f}(x)^n) = \mathrm{D}((x^2 + 3\,x - 5)^{(4/3)}) \;=$$
$$\frac{4}{3}\,(x^2 + 3\,x - 5)^{(1/3)}\,\mathrm{D}(x^2 + 3\,x - 5) =$$
$$\frac{4}{3}\,(x^2 + 3\,x - 5)^{(1/3)}\,(2\,x + 3)$$

Example 3.

$$\mathrm{D}(\mathrm{f}(x)^n) = n\,\mathrm{f}(x)^{(n-1)}\,\mathrm{D}(\mathrm{f}(x))$$
$$\mathrm{f}(x) = x^6 - 7\,x^5 + 8\,x^3 + 9\,x^2 + 1, \quad n = \frac{1}{3}$$

$$\mathrm{D}(\mathrm{f}(x)^n) = \mathrm{D}((x^6 - 7\,x^5 + 8\,x^3 + 9\,x^2 + 1)^{(1/3)}) \;=$$

$$\frac{1}{3}\,(x^6 - 7\,x^5 + 8\,x^3 + 9\,x^2 + 1)^{-(2/3)}\,\mathrm{D}(x^6 - 7\,x^5 + 8\,x^3 + 9\,x^2 + 1) =$$
$$\frac{1}{3}\,(x^6 - 7\,x^5 + 8\,x^3 + 9\,x^2 + 1)^{-(2/3)}\,(6\,x^5 - 35\,x^4 + 24\,x^2 + 18\,x)$$

Motivation for PowCR: Try Some Algebra and Rules You Already Know

Suppose you want to find : $\mathrm{D}((x^2+3\,x-5)^{20})$. *Based upon your knowledge of other rules you could first multiply out, but that would be tedious, since*

$$
\begin{aligned}
&(x^2+3\,x-5)^{20} = x^{40}+60\,x^{39}+1610\,x^{38}+25080\,x^{37}+243295\,x^{36}+1407672\,x^{35}\\
&+3258690\,x^{34}-15639660\,x^{33}-145490505\,x^{32}-295234920\,x^{31}+1353572844\,x^{30}\\
&+7851690720\,x^{29}+746643570\,x^{28}-86198131440\,x^{27}-136763632860\,x^{26}\\
&+611666823528\,x^{25}+1684642164045\,x^{24}-3308390998380\,x^{23}\\
&-12885143259490\,x^{22}+15955479224040\,x^{21}+73261609810801\,x^{20}\\
&-79777396120200\,x^{19}-322128581487250\,x^{18}+413548874797500\,x^{17}\\
&+1052901352528125\,x^{16}-1911458823525000\,x^{15}-2136931763437500\,x^{14}\\
&+6734229018750000\,x^{13}+291657644531250\,x^{12}-15335333437500000\,x^{11}\\
&+13218484804687500\,x^{10}+14415767578125000\,x^{9}-35520142822265625\,x^{8}\\
&+19091381835937500\,x^{7}+19889465332031250\,x^{6}-42958740234375000\,x^{5}\\
&+37123870849609375\,x^{4}-19134521484375000\,x^{3}+6141662597656250\,x^{2}\\
&-1144409179687500\,x+95367431640625
\end{aligned}
$$

Instead, you might try to use PR by first considering a simpler case : $\mathrm{D}((x^2+3\,x-5)^2)$

$$
\begin{aligned}
\textit{Then, by PR,}\quad \mathrm{D}((x^2+3\,x-5)^2) &= \mathrm{D}((x^2+3\,x-5)\;(x^2+3\,x-5))\\
&= \mathrm{D}(x^2+3\,x-5)\,(x^2+3\,x-5) \;+\; \mathrm{D}(x^2+3\,x-5)\,(x^2+3\,x-5)\\
&= 2(x^2+3\,x-5)\,\mathrm{D}(x^2+3\,x-5)\\
&= 2(x^2+3\,x-5)\,(2\,x+3)
\end{aligned}
$$

Let's see if this pattern persists when we try n = 3. By PR and the above, we have

$$
\begin{aligned}
\mathrm{D}((x^2+3\,x-5)^3) &= \mathrm{D}((x^2+3\,x-5)\;(x^2+3\,x-5)^2)\\
&= \mathrm{D}(x^2+3\,x-5)\,(x^2+3\,x-5)^2 \;+\; \mathrm{D}((x^2+3\,x-5)^2)\,(x^2+3\,x-5)\\
&= (2\,x+3)\,(x^2+3\,x-5)^2 \;+\; 2(x^2+3\,x-5)\,(2\,x+3)\,(x^2+3\,x-5)\\
&= (2\,x+3)\,(x^2+3\,x-5)^2 \;+\; 2(2\,x+3)\,(x^2+3\,x-5)^2\\
&= 3\;(x^2+3\,x-5)^2\,(2\,x+3)
\end{aligned}
$$

Then so far the pattern is :

$$
\begin{aligned}
\mathrm{D}((x^2+3\,x-5)^2) &= 2\;(x^2+3\,x-5)^{\,\mathit{1}}\,\mathrm{D}(x^2+3\,x-5)\\
\mathrm{D}((x^2+3\,x-5)^3) &= 3\;(x^2+3\,x-5)^2\,\mathrm{D}(x^2+3\,x-5)
\end{aligned}
$$

Apparently, we can extend this pattern to answer our original question, which is much less tedious!

$$
\begin{aligned}
\mathrm{D}((x^2+3\,x-5)^{20}) &= 20\;(x^2+3\,x-5)^{19}\,\mathrm{D}(x^2+3\,x-5)\\
&= 20\;(x^2+3\,x-5)^{19}\,(2\,x+3)
\end{aligned}
$$

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