Exponential Functions & Flow Curves Cheat Sheet

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Exponential function:

$$e^{t} = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \frac{1}{5040}t^7 + \ldots$$

Parametrization of the unit circle: $[\cos(t), \sin(t)]$

$$\cos(t) = 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \ldots$$

$$\sin(t) = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \ldots$$

Parametrization of the unit hyperbola: $[\cosh(t), \sinh(t)]$

$$\cosh(t) = 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 + \ldots = \frac{e^{t} + e^{-t}}{2}$$

$$\sinh(t) = t + \frac{1}{6}t^3 + \frac{1}{120}t^5 + \frac{1}{5040}t^7 + \ldots = \frac{e^{t} - e^{-t}}{2}$$

Complex exponential:

$$e^{it} = 1 + it - \frac{1}{2}t^2 - \frac{1}{6}it^3 + \frac{1}{24}t^4 + \frac{1}{120}it^5 - \frac{1}{720}t^6 - \frac{1}{5040}it^7 + \ldots$$

$$e^{it} = \left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \ldots\right) + i\left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \ldots\right)$$

$$e^{it} = \cos(t) + i \cdot \sin(t)$$

Example with $t = \pi$: $e^{i\pi} + 1 = 0$

Complex exponential for $a + bi$:

$$e^{(a+bi)t} = e^{at} \cdot e^{(bi)t}$$

$$= e^{at}\left(1 + i\cdot(bt) - \frac{1}{2}\cdot(bt)^2 - \frac{1}{6}i\cdot(bt)^3 + \frac{1}{24}\cdot(bt)^4 + \frac{1}{120}i\cdot(bt)^5 - \frac{1}{720}\cdot(bt)^6 + \ldots\right)$$

$$e^{(a+bi)t} = e^{at}(\cos(bt) + i\cdot\sin(bt))$$

Matrix exponential function for a square n-by-n matrix A:

$$e^{At} = I + A\,t + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \frac{1}{24}A^4t^4 + \frac{1}{120}A^5t^5 + \frac{1}{720}A^6t^6 + \frac{1}{5040}A^7t^7 + \ldots$$

where I is the n-by-n identity matrix.

Example for a diagonal matrix: let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and verify that $e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$ **(I)**

This will be very useful when we model the flow of liquid in a system of tanks.

Derivative of matrix exponential function for a square matrix n-by-n matrix A:

$$\mathrm{D}[\mathrm{e}^{At}] = \mathrm{D}\left[I + A\,t + \frac{1}{2}A^2 t^2 + \frac{1}{6}A^3 t^3 + \frac{1}{24}A^4 t^4 + \frac{1}{120}A^5 t^5 + \frac{1}{720}A^6 t^6 + \frac{1}{5040} + A^7 t^7 + \ldots \right]$$
$$= A + A^2 t + \frac{1}{2}A^3 t^2 + \frac{1}{6}A^4 t^3 + \frac{1}{24}A^5 t^4 + \frac{1}{120}A^6 t^5 + \frac{1}{720}A^7 t^6 + \ldots$$
$$= A\left(I + A\,t + \frac{1}{2}A^2 t^2 + \frac{1}{6}A^3 t^3 + \frac{1}{24}A^4 t^4 + \frac{1}{120}A^5 t^5 + \frac{1}{720}A^6 t^6 + \ldots \right)$$
$$\mathrm{D}[\mathrm{e}^{At}] = A\mathrm{e}^{At}$$

Matrices that capture the properties of the number 1 and the complex number *i*.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \textit{ matrix version of the number } 1,$$

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \textit{ matrix version of the number } i,$$

$$C^2 = C \bullet C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \textit{ matrix version of the number } -1 = i^2 = i \bullet i,$$

$$C^3 = C \bullet C^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \textit{ matrix version of the number } -i = i^3 = i \bullet i^2,$$

$$C^4 = C \bullet C^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \textit{ matrix version of the number } 1 = i^4 = i \bullet i^3,$$

$$a + bi \equiv a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \text{ complex numbers as 2-by-2 matrices.}$$

Recall the matrix exponential function (above) for a square n-by-n matrix A :

$$\mathrm{e}^{At} = I + A\,t + \frac{1}{2}A^2 t^2 + \frac{1}{6}A^3 t^3 + \frac{1}{24}A^4 t^4 + \frac{1}{120}A^5 t^5 + \frac{1}{720}A^6 t^6 + \frac{1}{5040}A^7 t^7 + \ldots$$

where I is the n-by-n identity matrix.

(1) Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and verify that $\mathrm{e}^{At} = \begin{bmatrix} \mathrm{e}^{at} & 0 \\ 0 & \mathrm{e}^{at} \end{bmatrix}$.

(2) Let $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$ and verify that $\mathrm{e}^{Bt} = \begin{bmatrix} \cos(b\,t) & -\sin(b\,t) \\ \sin(b\,t) & \cos(b\,t) \end{bmatrix}$.

(3) So, the matrix version of the complex number formula $e^{(a+bi)t} = e^{at}(\cos(bt) + i\cdot\sin(bt))$ is

$$e^{(A+B)t} = e^{At}e^{Bt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix}\begin{bmatrix} \cos(b\,t) & -\sin(b\,t) \\ \sin(b\,t) & \cos(b\,t) \end{bmatrix} = \begin{bmatrix} e^{at}\cos(b\,t) & -e^{at}\sin(b\,t) \\ e^{at}\sin(b\,t) & e^{at}\cos(b\,t) \end{bmatrix} \qquad \textbf{(II)}$$

where $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$, $A + B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and since $AB = BA$.

This will be very useful when we model the flow of liquid in a closed system of tanks.

A Linear Dynamical System in $\mathbb{R}^3$ is given by the following equations:

$\boldsymbol{\alpha}'(t) = A\cdot\boldsymbol{\alpha}(t)$, $\boldsymbol{\alpha}(0) = K_0$, where $\alpha'(t)$ is the velocity vector field of a curve $\alpha(t)$ in $\mathbb{R}^3$, A is a 3-by-3 real matrix, and K_0 is a 3-by-1 real vector of initial conditions for the system.

Define ***the flow of the system*** by $F(t) = e^{At}$, then a ***solution curve*** or ***flow curve*** of the system is given by $\alpha(t) = F(t)\cdot K_0 = e^{At}K_0$, since as we have seen above $F'(t) = \mathrm{D}[e^{At}] = Ae^{At}$, and so, $\alpha'(t) = F'(t)\cdot K_0 = Ae^{At}K_0 = A\cdot\alpha(t)$ and $\alpha(0) = K_0$. Hence, $\alpha(t) = e^{At}K_0$ is a solution to the linear dynamical system and it's not hard to show it is the unique solution.

The ability to explicitly compute a solution curve or flow curve of the system depends on the ease or difficulty of explicitly computing the flow of the system $F(t) = e^{At}$. The standard approach is to find the ***eigenvalues*** and ***eigenvectors*** of the matrix A. The eigenvalues are obtained by finding the roots of the ***characteristic equation*** of A: $det(A - \lambda I) = 0$. If λ is an eigenvalue of A of multiplicity *m*, then the corresponding eigenvectors are found by finding a basis for the null space of the operator $(A - \lambda I)^m$. In the case of a real 3-by-3 matrix A, three linearly independent eigenvectors are found and used to create the columns of an inveritble matrix P. Next, the flow of the system, $F(t) = e^{At}$, is adjusted as follows:

$$F(t) = e^{At} = P\cdot P^{-1}\cdot e^{At}\cdot P\cdot P^{-1} = P\cdot e^{(P^{-1}\cdot A\cdot P)t}\cdot P^{-1} = P\cdot e^{Et}\cdot P^{-1}, \text{ where } E = P^{-1}\cdot A\cdot P.$$

Hence, this equivalent form of the flow, $F(t) = P\cdot e^{Et}\cdot P^{-1}$, provides another equivalent form of the solution flow curve $\boldsymbol{\alpha}(t) = F(t)\cdot K_0 = P\cdot e^{Et}\cdot P^{-1}K_0$. In general, this form is much easier to explicitly compute, since e^{Et} is much easier to compute than e^{At}. For example, if the matrix A has three distinct real eigenvalues then e^{Et} takes the form of the example **(I)** above. If A has one real and two complex conjugate eigenvalues, then a 2-by-2 submatrix of E takes the form of the 2-by-2 matrix in equation **(II)** above. Finally, if A has repeated eigenvalues then sometimes the version of e^{Et} with exponentials along the diagonal as in example **(I)** will suffice, and sometimes yet another decomposition of e^{At} is necessary. Here is a sketch of this additional decomposition needed for repeated eigenvalues. We begin with a diagonal matrix E with all eigenvalues on the diagonal including repeated entries for repeated eigenvalues. Next, let $S = P\cdot E\cdot P^{-1}$ and let $N = A - S$. Suppose the multiplicity of the repeated eigenvalue is 2. Now, since it can be shown that $S\cdot N = N\cdot S$ and that $N^2 = 0$ (matrix), we have the following

$$F(t) = e^{At} = e^{(S+N)t} = e^{St}\cdot e^{Nt} = e^{(P\cdot E\cdot P^{-1})t}\cdot e^{Nt} = P\cdot e^{Et}P^{-1}\cdot e^{Nt} = P\cdot e^{Et}P^{-1}\cdot(I + Nt).$$

Again, we see that the last expression on the right is quite easy to explicitly compute.
Note that the accompanying Maple worksheet contains several interactive examples of each of these cases for 3-by-3 matrices and the code for computing them.

Brine Tank Cascade

Let brine tanks A, B, C be given of volumes 20, 40, 60, respectively, as in Figure 1.

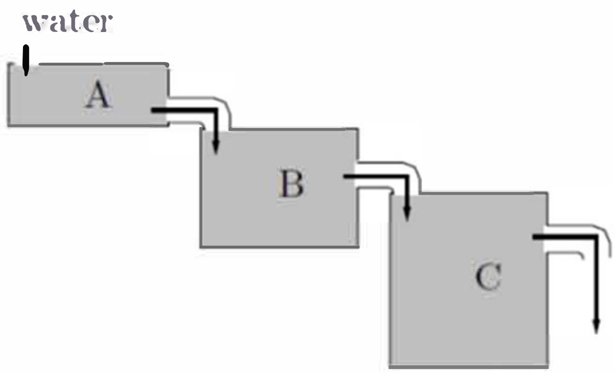

Figure 1. Three brine tanks in cascade.

It is supposed that fluid enters tank A at rate r, drains from A to B at rate r, drains from B to C at rate r, then drains from tank C at rate r. Hence the volumes of the tanks remain constant. Let $r = 10$, to illustrate the ideas.

Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.

Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t in each tank. We suppose added to tank A **water containing no salt**. Therefore, the salt in all the tanks is eventually lost from the drains. The cascade is modeled by the **chemical balance law** (see percent inflow/outflow discussion below).

Brine Tank Problem 1.

Find a curve $\alpha(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ whose velocity $\alpha'(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{bmatrix}$ models the flow through this system of brine tanks.

Let a, b, c be the percent outflow of liquid for tanks A, B, C, respectively. Then these values also give the percent outflow of brine for each (homogeneous) tank. In addition, they can be used to compute the inflow, outflow and net change of brine for each tank.

We have $a = \frac{10}{20} = \frac{1}{2}$, $b = \frac{10}{40} = \frac{1}{4}$ and $c = \frac{10}{60} = \frac{1}{6}$. So, each component of our velocity vector corresponds to the rate of change of brine in the corresponding tank.

$$\alpha'(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{bmatrix} = \begin{bmatrix} -a\,x_1(t) \\ a\,x_1(t) - b\,x_2(t) \\ b\,x_2(t) - c\,x_3(t) \end{bmatrix} = \begin{bmatrix} -a & 0 & 0 \\ a & -b & 0 \\ 0 & b & -c \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(1) Find the eigenvalues and eigenvectors of the coefficient matrix $A = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{6} \end{bmatrix}$.

(2) Using (1), let $L = P^{-1} A\, P$, let $K = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, compute $e^{Lt} = I + L\,t + \frac{1}{2} L^2 t^2 + \frac{1}{6} L^3 t^3 + \ldots$

and finally, set $\alpha(t) = Pe^{Lt}K$. (guess this possible solution)

(3) Using the $\alpha(t)$ created in (2), show that $\alpha'(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = A\alpha(t)$, i.e. the velocity

vector for this curve models the flow through this system of brine tanks.

(4) Using (2) and (3), if you evaluate the solution $\alpha(t)$ at $t = 0$, then the components of the vector $\alpha(0)$ should correspond to the initial quantity of brine in each tank. However, as you can see (calculate), these values are rather odd using the vector K in (2). Find a vector K such that both the initial quantity of brine in each tank at $t = 0$ is 20 percent of the volume of each tank and such that the formula for the solution in (2) yields the vector with these initial conditions.